# Game Theory - Part 1

Mar. 27, 2025

# Recap question:

March 27, 2025

Bob has computed the multiplicative inverse of 2 modulo 7 and written it down in his HW solution. The day after computing it, he realizes he cannot read his hand-writing. He thinks he has written either a 7, a 5 or a 4. Which one is the correct multiplicative inverse?

# Recap question:

## March 27, 2025

We can try them in turn and see if 2 times each number is equivalent to 1 modulo 7. We have

$$7 \times 2 = 14 \equiv 0 \pmod 7$$
$$5 \times 2 = 10 \equiv 3 \pmod 7$$
$$4 \times 2 = 8 \equiv 1 \pmod 7$$

This means that 4 is the multiplicative inverse of 2 modulo 7. Alternatively, we could use the Euclidean algorithm and compute the multiplicative inverse straight away.

# Game Theory - Part 1

March 27, 2025

By the end of this lecture, you will be able to:

1. Give examples of applications of game theory
2. Identify equilibrium points of games

**Game theory** is the study of strategic interaction among several decision makers. The goal is to provide a common language to describe and understand seemingly different scenarios.

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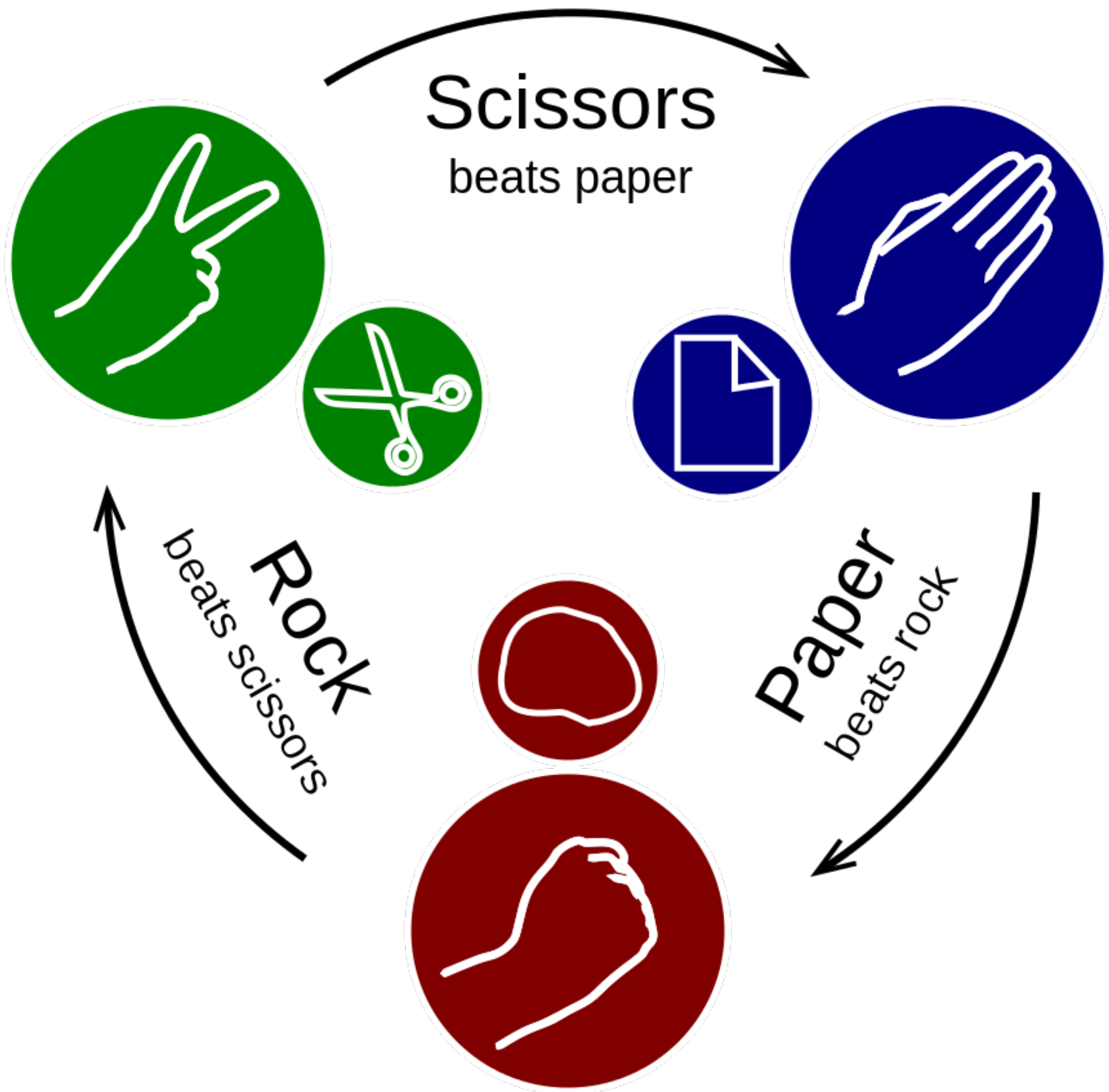
In this unit we will analyze games, and determine how to make a decision based on the strategies of other players.

# **Example 1:** actual games




https://en.wikipedia.org/wiki/File:Rock-paper-scissors.svg


https://commons.wikimedia.org/wiki/File:Ivory_Coast_penalty.jpg

# **Example 1:** actual games

## The price of anarchy in basketball

Brian Skinner
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(Dated: November 8, 2011)

Optimizing the performance of a basketball offense may be viewed as a network problem, wherein each play represents a "pathway" through which the ball and players may move from origin (the inbounds pass) to goal (the basket). Effective field goal percentages from the resulting shot attempts can be used to characterize the efficiency of each pathway. Inspired by recent discussions of the "price of anarchy" in traffic networks, this paper makes a formal analogy between a basketball offense and a simplified traffic network. The analysis suggests that there may be a significant difference between taking the highest-percentage shot each time down the court and playing the most efficient possible game. There may also be an analogue of Braess's Paradox in basketball, such that removing a key player from a team can result in the improvement of the team's offensive efficiency.

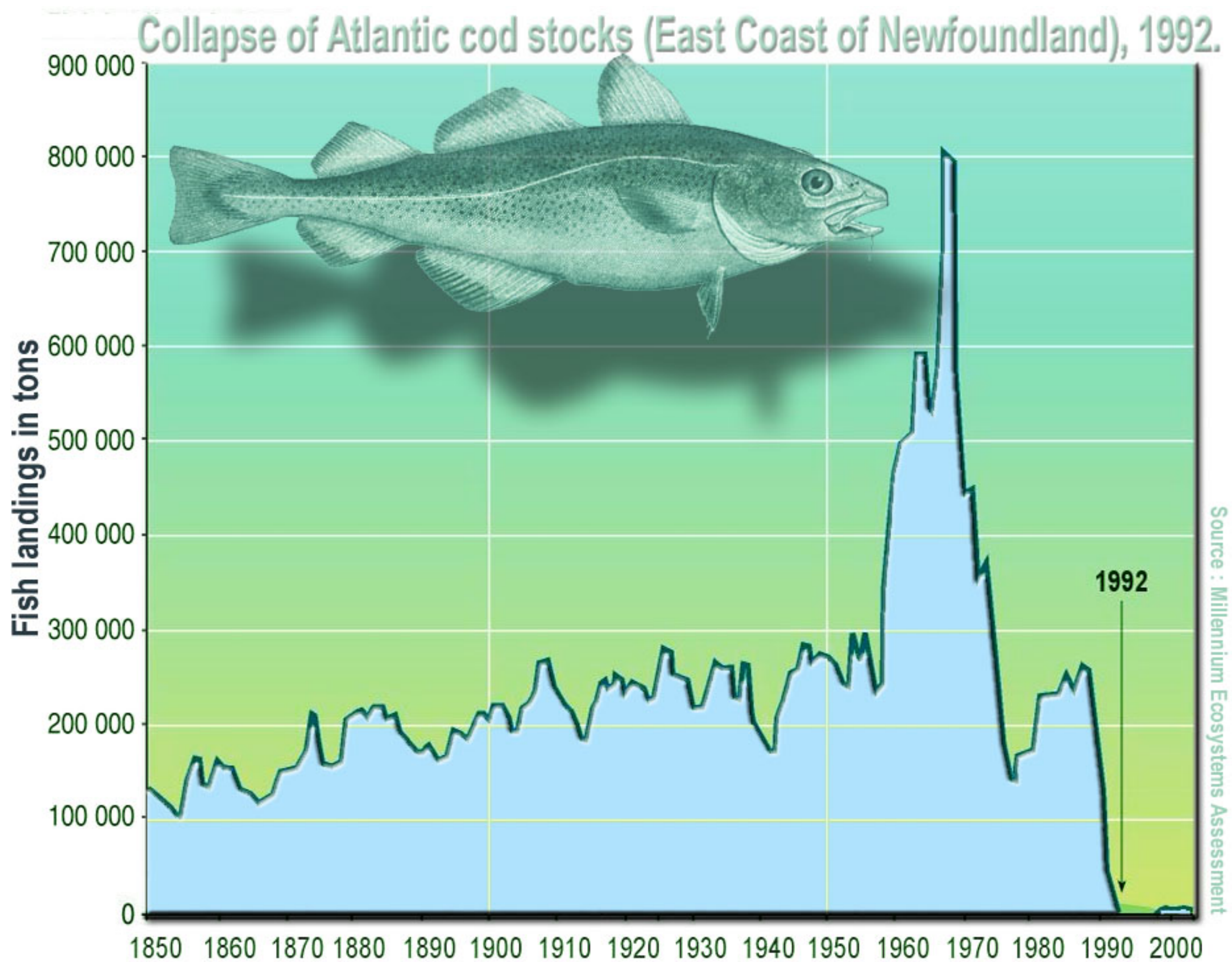
https://en.wikipedia.org/wiki/File:Rock-paper-scissors.svg https://commons.wikimedia.org/wiki/File:Ivory_Coast_penalty.jpg

**Example 2:** negotiations

# **Example 3:** biology

**Figure 2. (A)** Meerkats foraging in Kalahari Gemsbok Park. **(B)** Individuals commonly dig up to 20 cm below ground to find prey and cannot watch for predators while doing so. **(C)** Digging animals frequently stop to glance around them but do so less frequently if a raised guard is vocalizing. **(D)** While the group is foraging, one animal commonly climbs a tree or mound and watches for predators. (Clutton Brock et al. 1999b)

**Example 4:** most examples of overfishing follow the same pattern. A new fishing site is discovered with plenty of fish, which attracts many fishermen. Wanting to make extra profit, they fish as much as they can as quickly as they can, which does not give the system time to recover.



**Example 4:** most examples of overfishing follow the same pattern. A new fishing site is discovered with plenty of fish, which attracts many fishermen. Wanting to make extra profit, they fish as much as they can as quickly as they can, which does not give the system time to recover. The result is that the site becomes depleted. Rather than earning more money, the fishing strategy is contrary to the common good and ruins all the fishermen.

Compare to climate change. How do we design incentives that avoid overfishing/global warming?

**Example 5:** residencies for medical students in the 1940s. An early offer from a hospital with a short deadline to consider increases chances students will accept. Hospitals competed in giving earlier and earlier offers which led to students being hired years before finishing their studies, i.e. before it is possible to tell if they will be good doctors or not. How do we design a better system?

**Goal of today:** develop a mathematical language and a way to reason about seemingly different situations that share common features.

# Definition of a game

1. A game has several decision-makers (players).

2. Each player has their own interests. The interests of different players may or may not conflict.

3. Each player has several strategies to choose from.

4. The outcome for each player depends on the decisions of all the players.

# Definition of a game

We will assume that each player is **rational**, which means that the player wants to choose the strategy that leads to the best outcome for them (expressed through the payoff).

The challenge is that the outcome for each player depends on the decisions of all players, not just their own.

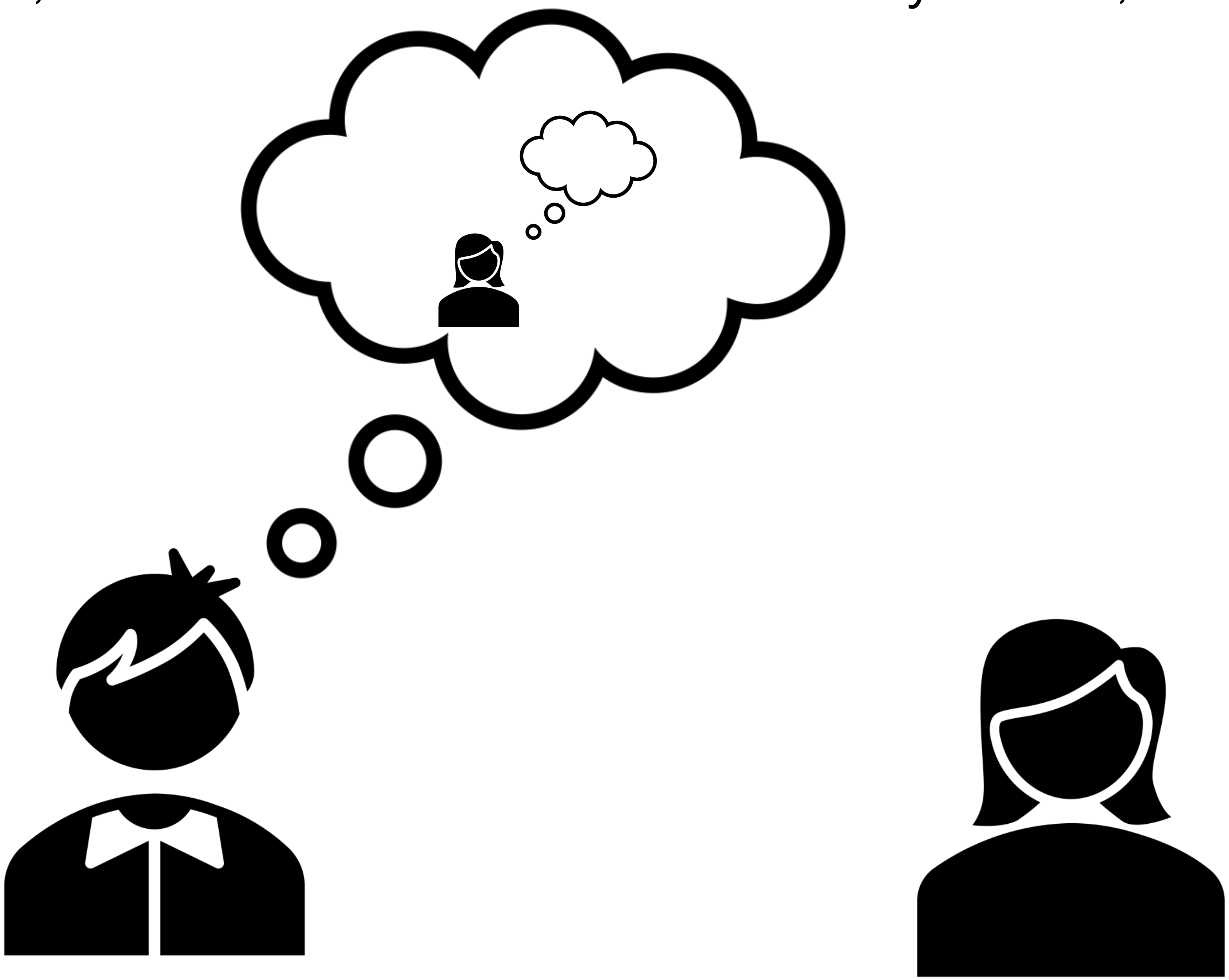
Today, we assume

1. Each player knows all possible strategies of the other players.

2. All players make their decision simultaneously.

3. All players are rational so they make decisions so as to maximize their payoffs.

We will think a lot about other players' strategies,
i.e., we will think about what they know,



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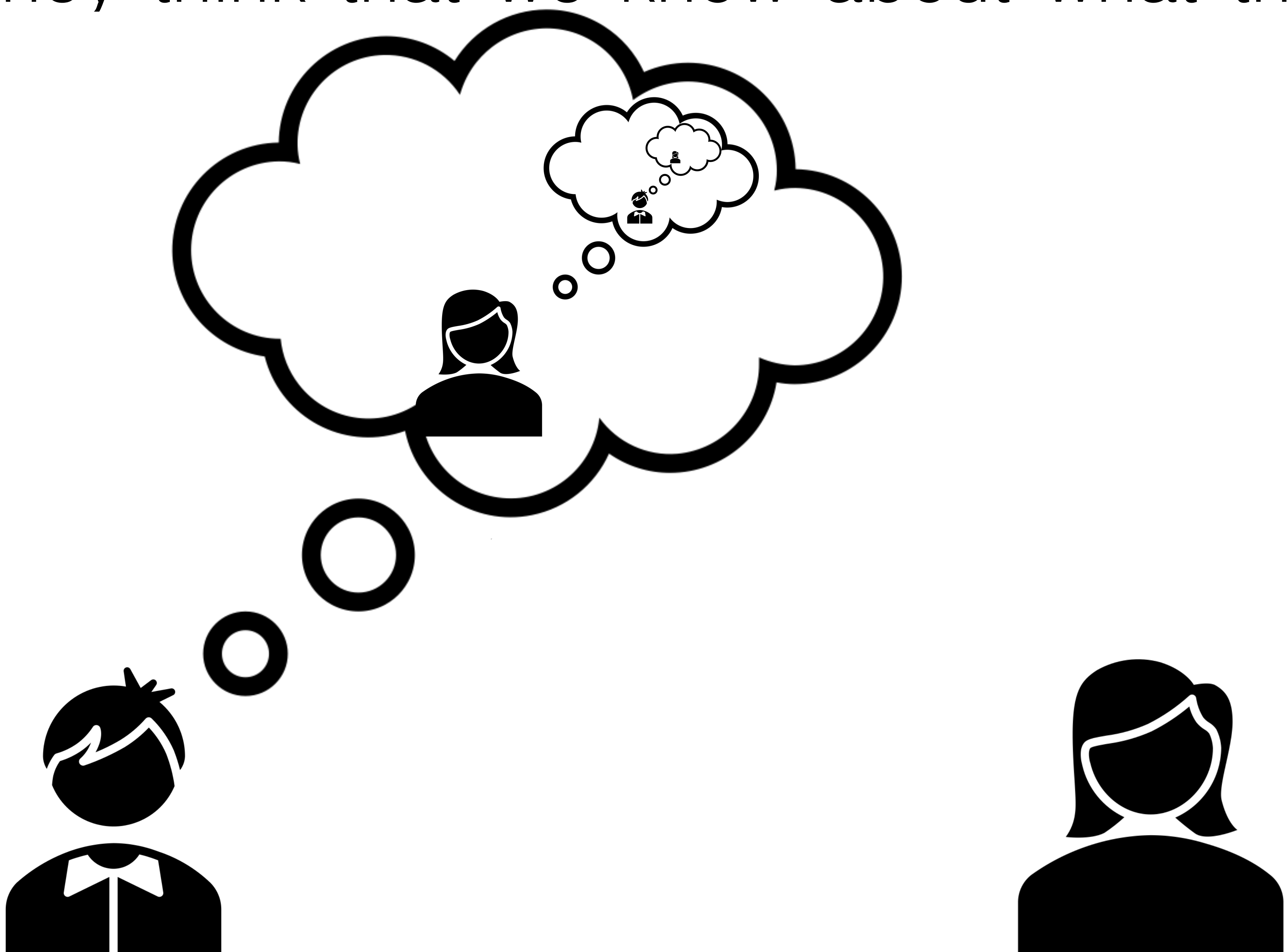

We will think a lot about other players' strategies, i.e., we will think about what they know, and what they think that we know about what they know, and what they think that we know about what they think that we know about what they know, etc.

To illustrate this, we start with a game and a puzzle.

# A game

On a piece of paper, write down your name, and an integer from 0 to 100

The winner is the one whose number is closest to 2/3 of the average of the integers you collectively picked

**Reward:**
The winner gets one point added to their score on HW 4 (up to a max of 50 points).

# A game: results of round 1

2/3 of the mean is ?
Closest guess is ?

# A game: results of round 2

2/3 of the mean is ?
Closest guess is ?

# Anticipating opponent strategies for the game

The best strategy is to guess a number 2/3 of the average of what we think the others will guess. If everyone guesses like this, then every round we play the game, the average should decrease.

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After a while, the best guess should be 0. If everyone guesses 0, this is an equilibrium point: no single person benefits from changing their guess away from 0.

However, if you have a good reason to believe that not everyone else is thinking along these lines, then you would benefit from choosing something other than 0.

# A puzzle

A game show puts a group of 20 people on an island and place a red hat on 17 people and a blue hat on 3 people. They are not allowed to take off their hats, discuss their hat color with each other or look in any mirrors.

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On the third day, someone leaves on the ferry. How could they be sure they had a blue hat?

Hint: if there was only one person with a blue hat, does anyone leave on the first day? If there were only 2 people with blue hats, does anyone leave on the first day? If not, what does the fact that no-one leaves on the first day tell someone wearing a blue hat? Same questions for the second day.

# A puzzle

**Solution:**

On the third day, all 3 people wearing blue hats leave on the ferry.

# A puzzle

What would happen if there were only one person with a blue hat? That person sees that no-one else has a blue hat and realizes that she must be the person wearing blue, and thus leaves on the ferry the first day.

# A puzzle

If there were two people wearing blue hats, each of these sees one other person wearing a blue hat. They would reason as follows: “If I am not wearing a blue hat, then there would only be 1 blue-hatted person on this island, and so they will leave on day 1”. But when 1 day passes, none of the blue-hatted people leave (because at that stage they have no evidence that they themselves are blue-hatted).

# A puzzle

After nobody leaves on the 1st day, each of the blue-hatted people then realizes that they themselves must have a blue hat, and will then leave on the 2nd day.

# A puzzle

If there are 3 people with blue hats, they each see 2 others with blue hats. Each blue-hatted person would reason as follows: “If I am not blue-hatted, then there will only be 2 blue-hatted people on this island, and so they will all leave on the 2nd day”. But when the 2nd day passes, none of the blue-hatted people leave (because at that stage they have no evidence that they themselves are blue-hatted).

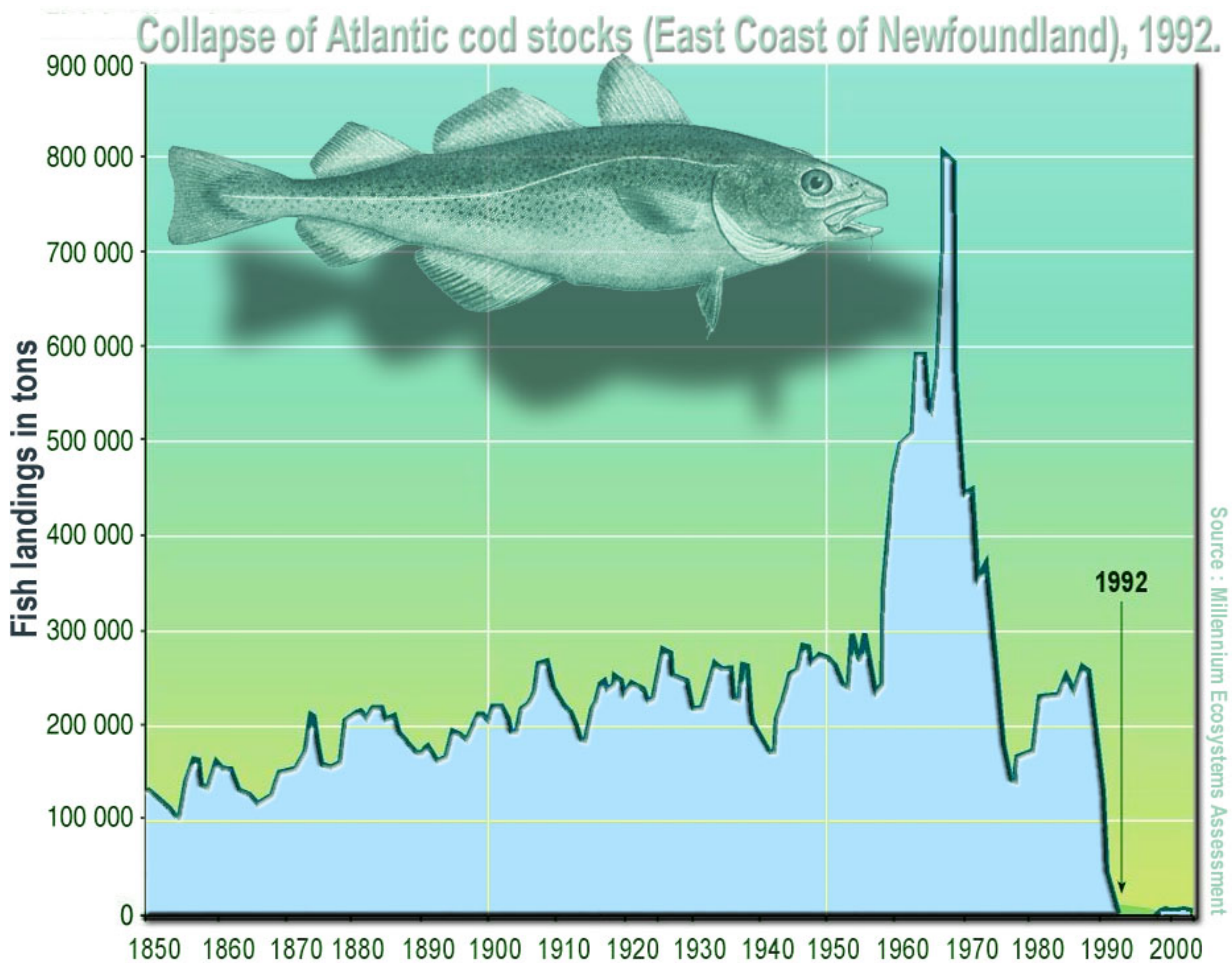
# A puzzle

After nobody leaves on the 2nd day, each of the 3 blue-hatted people then realizes that they themselves must have a blue hat, and will then leave on the 3rd day.

**Takeaway:** When we know the other players' possible strategies and payoffs, and they know that we know, and we know that they know that we know, and so on, we can come to conclusions about their actions (from their strategies and payoffs and from their knowledge of our strategies and payoffs and their knowledge that we know they know and so on).

# Anticipating opponent strategies when fishing

**Example:** overfishing again. If every fisherman is expecting every other fisherman to fish as much as possible, their best strategy is to fish as much as possible.



# Describing games

We will use the following kinds of diagrams (called a payoff matrix) to describe games. This game has two players with two strategies each:

| Player A | | player B: strategy 1 | player B: strategy 2 |
|---|---|---|---|
| | strategy 1 | (5, -1) | (2, 9) |
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Reward obtained by player A if player A picks strategy 1 and player B picks strategy 1

Reward obtained by player B if player A picks strategy 1 and player B picks strategy 1

**Example 1:** Is rock-paper-scissors a game according to our definition?

1. There are 2 players.

2. Each player has 3 strategies (rock, paper, scissors).

3. The outcome depends on the actions of both players.

We define the payoff of a draw to be 0, the payoff of a victory to be 5 dollars and the payoff of a loss to be $-5$ dollars.

We define the payoff of a draw to be 0, the payoff of a victory to be 5 dollars and the payoff of a loss to be –5 dollars.

| Player A \ player B | Rock | Paper | Scissors |
|---|---|---|---|
| Rock | (0, 0) | (-5, 5) | (5, -5) |
| Paper | (5, -5) | (0, 0) | (-5, 5) |
| Scissors | (-5, 5) | (5, -5) | (0, 0) |

**Example 2:** penalty kicks in football. The player can kick the ball either to the left or the right and the goalie can dive either to left or right to catch it. If the ball goes in, the kicking team gets 1 point.

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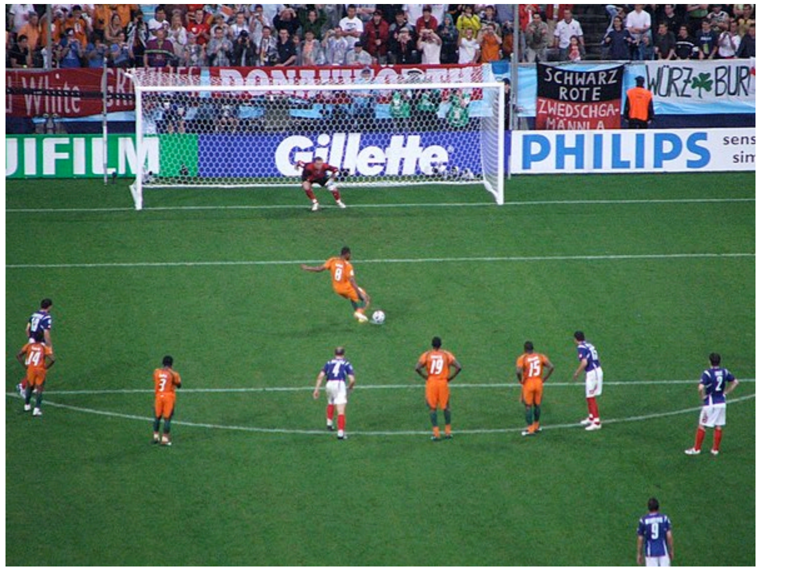


**Goalie**

| **Kicker** | Dive left | Dive right |
|---|---|---|
| Kick left | $(0, 0)$ | $(1, 0)$ |
| Kick right | $(1, 0)$ | $(0, 0)$ |