Game Theory - Part 2

April 1, 2025

Recap question:

April 1, 2025

You are player A in the game described by the following payoff matrix. If player B plays strategy 2, what strategy would be best for you to play? What if player B plays strategy 1?

Player B

Player A	Strategy 1	Strategy 2
Strategy 1	(2,1)	(1,0)
Strategy 2	(1,10)	(-3,1)

Recap question:

April 1, 2025

The answer to both is strategy 1. If player B plays strategy 2, we get reward 1 if we play strategy 1 and reward -3 if we play strategy 2

Player B

Player A	Strategy 1	Strategy 2
Strategy 1	(2,1)	(1,0)
Strategy 2	(1,10)	(-3,1)

If player B plays strategy 1, we get reward 2 if we play strategy 1 and reward 1 if we play strategy 2

Player B

Player A	Strategy 1	Strategy 2
Strategy 1	(2,1)	(1,0)
Strategy 2	(1,10)	(-3,1)

Game Theory - Part 2

April 1, 2025

By the end of this lecture, you will be able to:

1. Solve sequential games using backward induction
2. Define Nash equilibria
3. Define dominating strategies of games
4. Define pure and mixed strategies

A sequential game: 21 Flags

Sequential game = players do not make their moves at the same time.

There are 21 flags, and two players alternate in taking turns to remove some flags. At each turn, a player can remove 1, 2, or 3 flags. The player who removes the last flag (whether as the sole remaining flag or one of the last surviving set of 2 or 3 flags) is the winner.

21 Flags

We fill in the following table. Green color for integer n means that the player to move when n flags remain will win. Red means that the player to move loses

										$\dots$
n	1	2	3	4	5	6	7	8	9	10

21 Flags

We fill in the following table. Green color for integer n means that the player to move when n flags remain will win. Red means that the player to move loses

										$\cdots$
n	1	2	3	4	5	6	7	8	9	10

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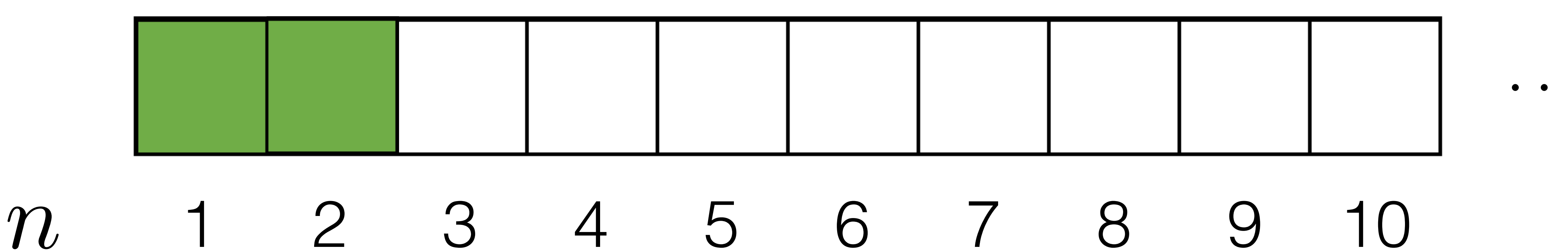

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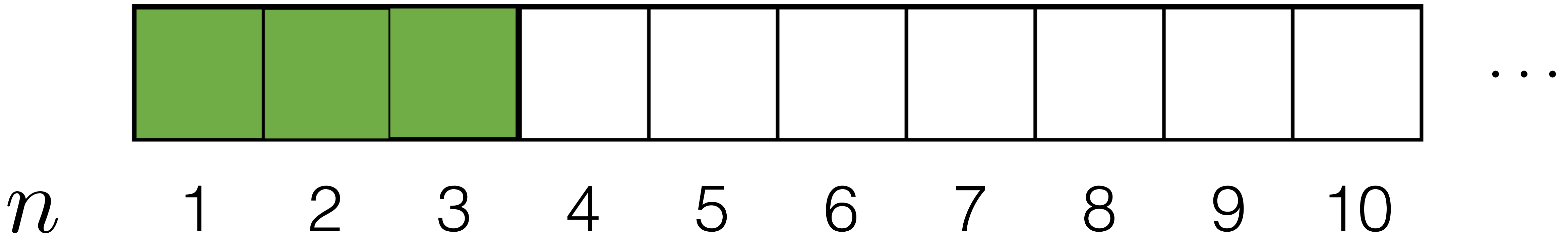

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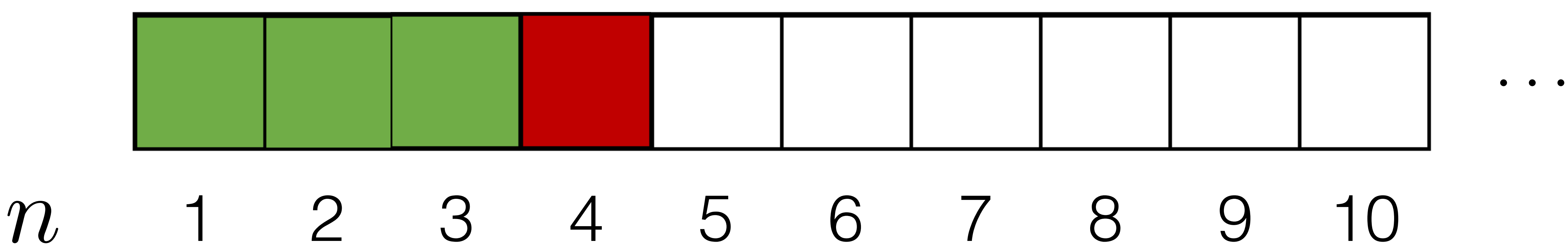

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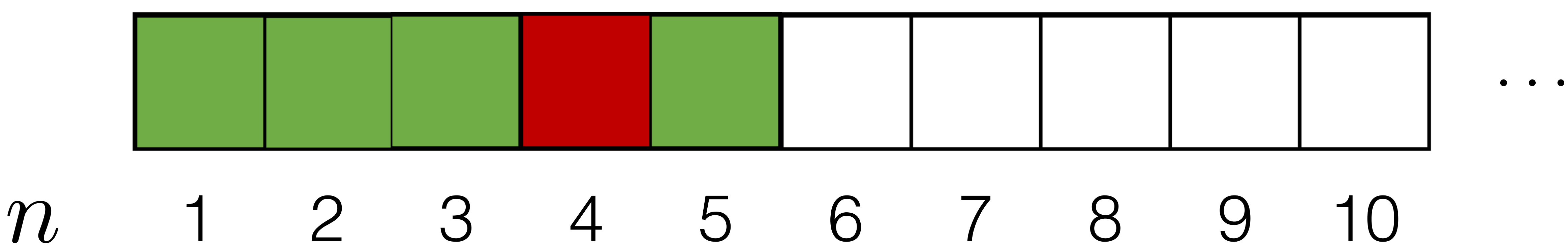

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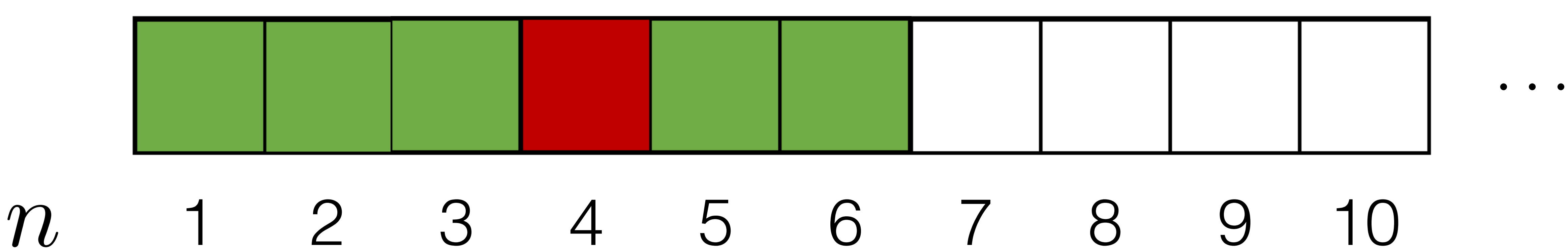

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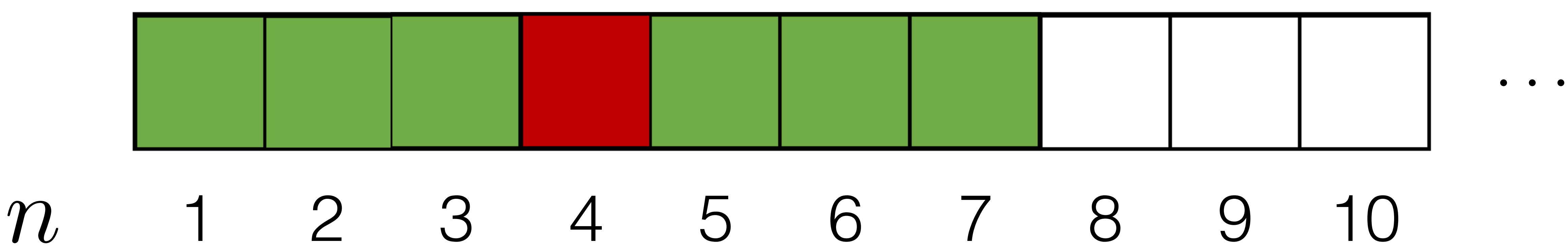

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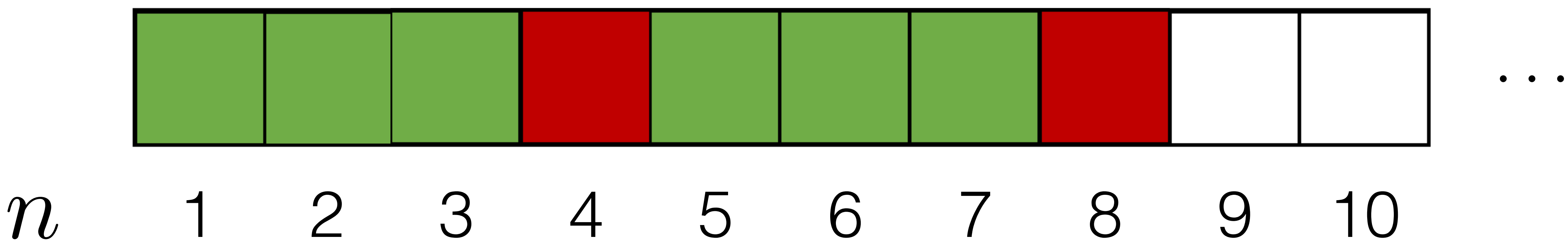

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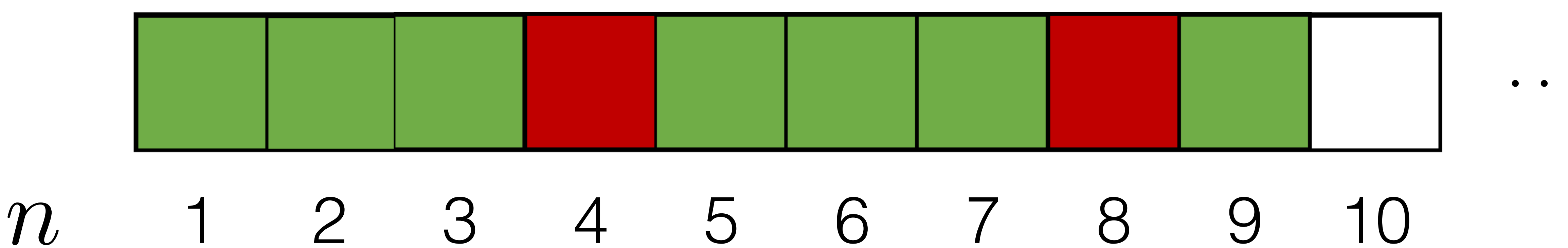

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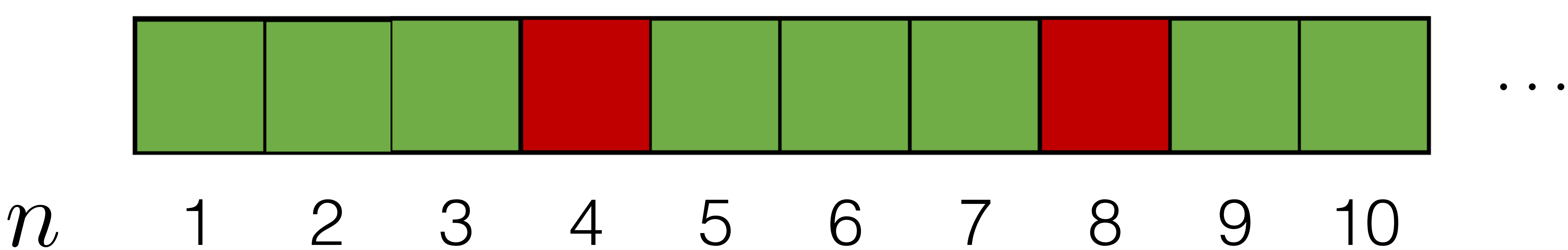


21 Flags

How to change the number “21” so that the second player always wins (with optimal play)?

21 Flags

How to change the number “21” so that the second player always wins (with optimal play)?

Replace 21 by 4, 8, 12, 16 or 20.

21 Flags

This process is called **backward induction:** start at the ending position and work your way backwards to any starting position, classifying each position as a win or a lose. This can in principle be applied to any sequential game. However, for some games (chess, go, …) there are too many possibilities to consider, so it becomes computationally infeasible.

Back to non-sequential games

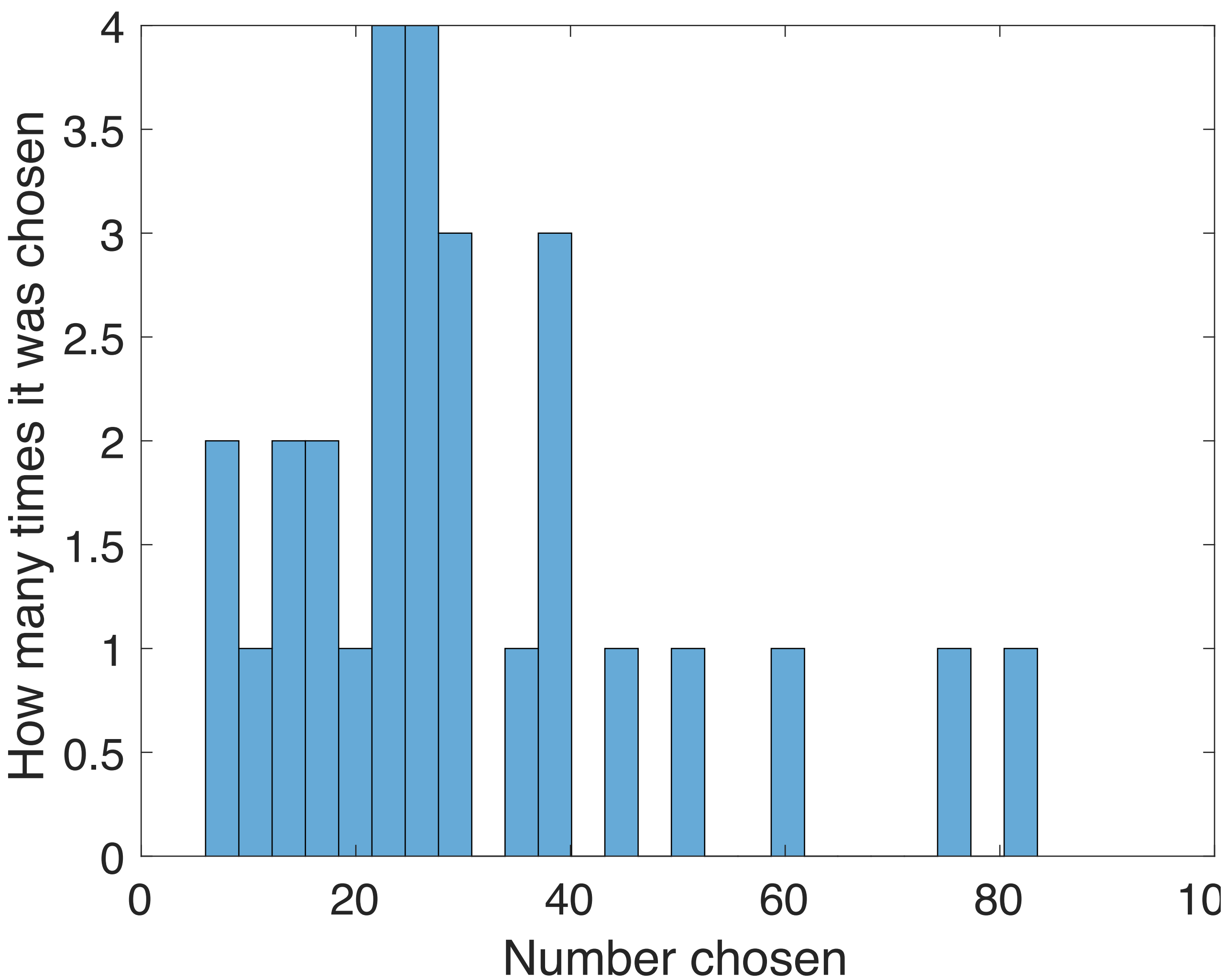
Back to non-sequential games

Last time, we played the following game:

"On a piece of paper, write down your name, and an integer from 0 to 100.

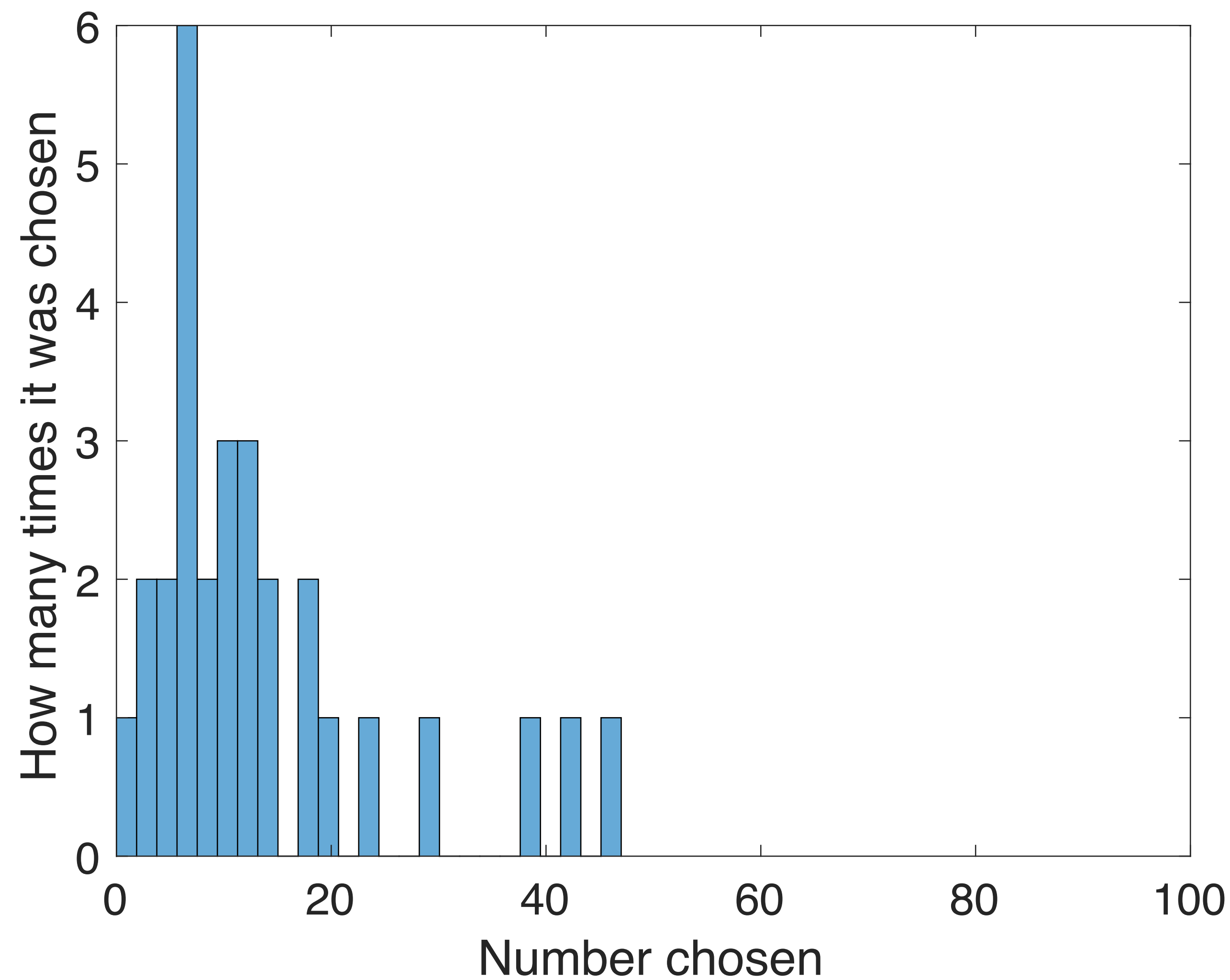
The winner is the one whose number is closest to 2/3 of the average of the integers you collectively picked"

Results of round 1



2/3 of the mean is 20.4
Closest guess is 19

Results of round 2



2/3 of the mean is 9.4
Closest guess is 9

Anticipating opponent strategies for the game

The best strategy is to guess a number 2/3 of the average of what we think the others will guess. If everyone guesses like this, then every round we play the game, the average should decrease.

Anticipating opponent strategies for the game

The best strategy is to guess a number 2/3 of the average of what we think the others will guess. If everyone guesses like this, then every round we play the game, the average should decrease.

After a while, the best guess should be 0. If everyone guesses 0, this is an equilibrium point: no single person benefits from changing their guess away from 0.

Anticipating opponent strategies for the game

The best strategy is to guess a number 2/3 of the average of what we think the others will guess. If everyone guesses like this, then every round we play the game, the average should decrease.

After a while, the best guess should be 0. If everyone guesses 0, this is an equilibrium point: no single person benefits from changing their guess away from 0.

However, if you have a good reason to believe that not everyone else is thinking along these lines, then you would benefit from choosing something other than 0.

Anticipating opponent strategies for the game

There is no reason to give a non-zero answer if you believe that others are absolutely rational.

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There is no reason to give a non-zero answer if you believe that others are absolutely rational.

The strategy that everyone answers 0 is a **Nash equilibrium.**

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There is no reason to give a non-zero answer if you believe that others are absolutely rational.

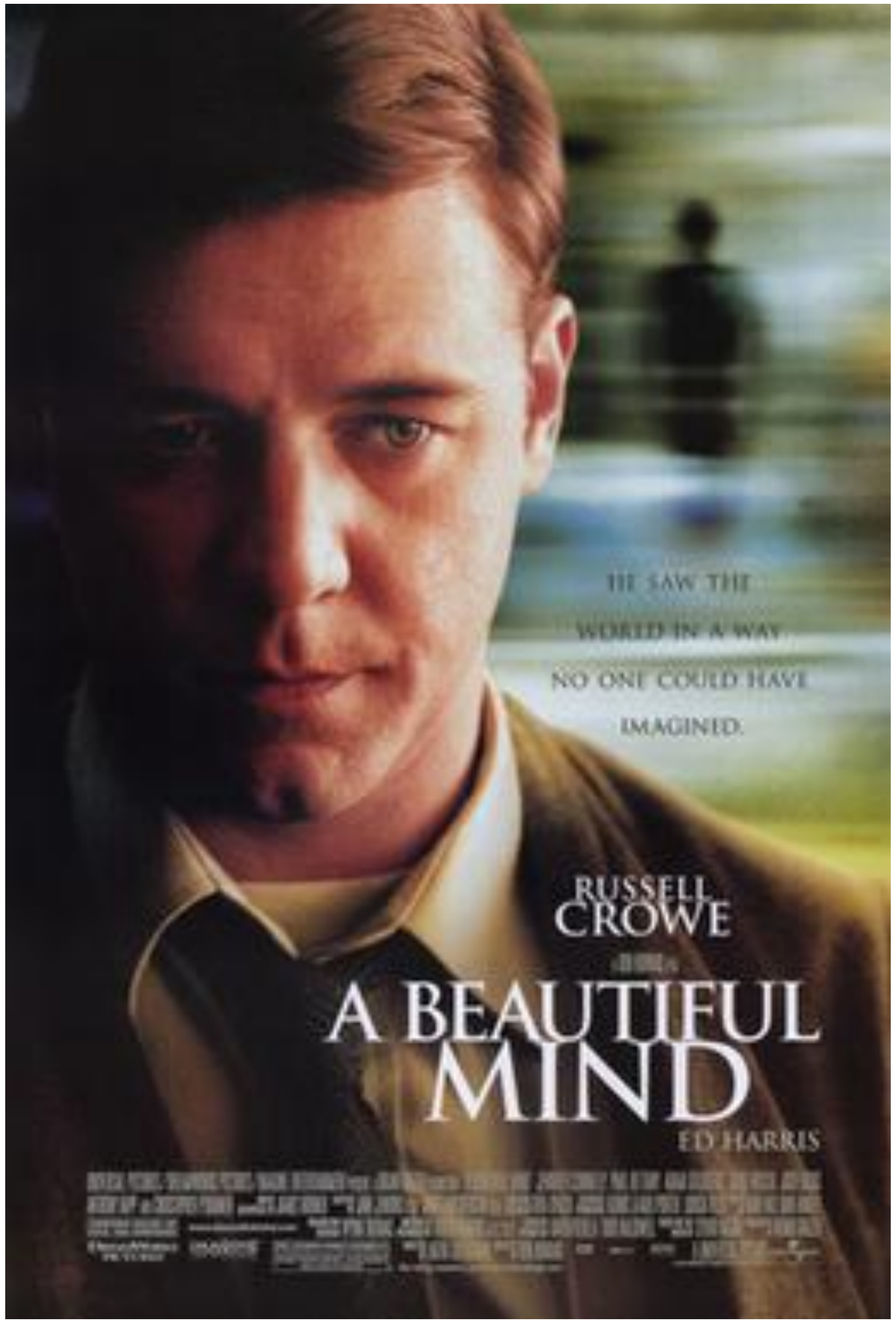
The strategy that everyone answers 0 is a **Nash equilibrium.**

A strategy is a Nash equilibrium if each player has chosen a strategy and no player can benefit by changing his/her strategy while the other players keep theirs unchanged.


If they're not taking care of it, I'm not doing it either

Anticipating opponent strategies for the game

In many applications, players (eventually) tend to play the Nash equilibrium, either from learning by playing the game multiple times or by analyzing it.



A Beautiful Mind is a 2001 American biographical drama film based on the life of the American mathematician **John Nash**, a Nobel Laureate in Economics and Abel Prize winner.

Discussion

Is a Nash equilibrium always unique?

Discussion

There can be multiple Nash equilibria. For the game with “2/3” replaced by “1”, the strategy where everyone guesses e.g. 50 is a Nash equilibrium, because everyone is then closest to the average and no-one benefits from changing their guess.

Discussion

Do Nash equilibria always exist?

Discussion

Do Nash equilibria always exist?

		player B		
		Rock	Paper	Scissors
Player A	Rock	(0, 0)	(-5, 5)	(5, -5)
	Paper	(5, -5)	(0, 0)	(-5, 5)
	Scissors	(-5, 5)	(5, -5)	(0, 0)

Discussion

Do Nash equilibria always exist?

		player B		
		Rock	Paper	Scissors
Player A	Rock	(0, 0)	(-5, 5)	(5, -5)
	Paper	(5, -5)	(0, 0)	(-5, 5)
	Scissors	(-5, 5)	(5, -5)	(0, 0)

Rock-paper-scissors does not have a Nash equilibrium in the way we have discussed so far (but it does when we expand what we mean by strategies at the end of the lecture).

Discussion

Is a Nash equilibrium always the best choice?

Discussion

If one has a good reason to believe that others will not be playing their Nash equilibrium strategies, then one's optimal choice will also differ from one's own Nash equilibrium strategy.

Discussion

What if we believe that others will be playing their Nash equilibrium strategies, is the Nash equilibrium always the optimal choice?

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What if we believe that others will be playing their Nash equilibrium strategies, is the Nash equilibrium always the optimal choice?

If everyone else is playing their Nash equilibrium, then the definition of a Nash equilibrium precisely means that we do best by playing the Nash equilibrium ourselves.

A game

On a piece of paper, write down your name, and a letter, either A or B.

If everyone chooses A, I will add 0.5 points to everyone's score on HW 5. If everyone chooses B, I will add 0.75 points for everyone. However, if both A and B are selected by at least one student, I will add 1 points to students choosing A and 0 points to students choosing B.

Discussion

“A strategy is a Nash equilibrium if each player has chosen a strategy and no player can benefit by changing his/her strategy while the other players keep theirs unchanged.”

Which strategy is the Nash equilibrium?

Does it lead to the maximal payoff for everyone?

The case of two students

		Player 2	
		A	**B**
Player 1	**A**	$(0.5, 0.5)$	$(1,0)$
	B	$(0,1)$	$(0.75,0.75)$

The case of two students

Nash equilibrium

Player 2

Player 1		A	B
	A	(0.5, 0.5)	(1,0)
	B	(0,1)	(0.75,0.75)

The case of two students

Nash equilibrium

Player 2

Player 1		A	B
	A	(0.5, 0.5)	(1,0)
	B	(0,1)	(0.75,0.75)

Strategy that leads to maximal payoff for everyone

The case of two students

		Player 2	
		A	**B**
Player 1	**A**	$(0.5,\ 0.5)$	$(1,0)$
	B	$(0,1)$	$(0.75,0.75)$

Strategy A is not just a Nash-equilibrium strategy…

The case of two students

		Player 2 A	Player 2 B
Player 1	A	$(0.5, 0.5)$	$(1,0)$
	B	$(0,1)$	$(0.75,0.75)$

A strategy of one player is a **dominant** strategy if the profit from this strategy is the highest regardless the strategy of other players.

The case of two students

		Player 2: A	Player 2: B
Player 1	A	$(0.5, 0.5)$	$(1,0)$
	B	$(0,1)$	$(0.75,0.75)$

A strategy of one player is a **dominant** strategy if the profit from this strategy is the highest regardless the strategy of other players.

Which one (A or B) is dominant?

The case of two students

		Player 2: A	Player 2: B
Player 1	A	(0.5, 0.5)	(1,0)
	B	(0,1)	(0.75,0.75)

A strategy of one player is a **dominant** strategy if the profit from this strategy is the highest regardless the strategy of other players.

Which one (A or B) is dominant? A is dominant since if player 2 plays B, A is better for player 1, and same if player 2 plays A.

A Nash equilibrium may not be the optimal solution.

Decision-making driven by personal interests may not lead to the optimal choice for the whole society.

When designing games (e.g. as a lawmaker), ensure that the Nash equilibria are beneficial to the entire community.

Prisoner's dilemma

Two (rational) suspected robbers are interrogated by the police. If one suspect confesses (defects) and the second denies (cooperates) then the suspect that defected will not serve jail time and the second suspect will serve 10 years. If both suspects defect they will each serve 5 years. If both suspects cooperate they will each serve 2 years

Player 1		player 2: defect	player 2: cooperate
	defect	(-5, -5)	(0,-10)
	cooperate	(-10, 0)	(-2, -2)

Which strategy is the Nash equilibrium?

Which strategy is the best for both of them?

Nash equilibrium

Player 1	player 2: defect	player 2: cooperate
defect	(-5, -5)	(0,-10)
cooperate	(-10, 0)	(-2, -2)

Strategy that is best for both of them

A modified game

On a piece of paper, write down your name, and a letter, either A or B.

If everyone chooses B, I will add 1.25 points for everyone on HW 5. However, if both A and B are selected by some students, I will add 1 point to students choosing A and 0 points to students choosing B.

Discussion

What are the Nash equilibria?

Was your answer in the modified game different from your answer in the previous game?

The case of two students

		Player 2	
		A	**B**
Player 1	**A**	$(1, 1)$	$(1,0)$
	B	$(0,1)$	$(1.25,1.25)$

The case of two students

Nash equilibrium

Player 2

Player 1		A	B
	A	$(1, 1)$	$(1,0)$
	B	$(0,1)$	$(1.25,1.25)$

Nash equilibrium

Example: Does the following game have any pure-strategy Nash equilibria?

Player 1 \ player 2	b_1	b_2	b_3	b_4
a_1	(3,10)	(4, -7)	(8, 20)	(-3, 8)
a_2	(8, 50)	(-9, -10)	(40, 30)	(0, 0)
a_3	(-8, 5)	(-3, 5)	(4, 5)	(9, 5)

Example: Does the following game have any pure-strategy Nash equilibria?

Player 1 \ player 2	b_1	b_2	b_3	b_4
a_1	(3,10)	(4, -7)	(8, 20)	(-3, 8)
a_2	(8, 50)	(-9, -10)	(40, 30)	(0, 0)
a_3	(-8, 5)	(-3, 5)	(4, 5)	(9, 5)