Game Theory - Part 3

April 3, 2025

Recap question:

April 3, 2025

You are player A in the game described by the following payoff matrix. What are the Nash equilibria of the game?

Player B

Player A

	Strategy 1	Strategy 2
Strategy 1	(2,1)	(1,0)
Strategy 2	(1,0)	(-3,1)

Recap question:

Top right: no, because player B would want to change to strategy 1.

Bottom right: no, because player A would want to change to strategy 1.

Bottom left: no, because player A would want to change to strategy 1.

Top left: this is a Nash equilibrium because neither player would want to change, if the other keeps their choice fixed.

	Strategy 1	Strategy 2
Strategy 1	(2,1)	(1,0)
Strategy 2	(1,0)	(-3,1)

Game Theory - Part 3

April 3, 2025

By the end of this lecture, you will be able to:

1. Define pure and mixed strategies
2. Find Nash equilibria of mixed strategies
3. Explain Braess's paradox

Mixed strategies

So far, all the strategies we have seen have been **pure strategies**, where we make up our mind and choose one strategy.

Mixed strategies

So far, all the strategies we have seen have been **pure strategies**, where we make up our mind and choose one strategy.

A **mixed strategy** is a mixture of several strategies, where each strategy is selected with some probability, in every round of the game. The sum of these probabilities is one for any mixed strategy.

Mixed strategies

Player one may have pure strategies a_1 and a_2 and a mixed strategy $0.3a_1 + 0.7a_2$. This means that, in the mixed strategy, player 1 selects the pure strategy a_1 with probability 0.3 and the pure strategy a_2 with probability 0.7.

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The average rewards we expect when playing the game many times are computed as a weighted average over the rewards of the pure strategies.

Mixed strategies

		player 2	
		Confess	Deny
Player 1	Confess	(-5, -5)	(0,-10)
	Deny	(-10, 0)	(-2, -2)

Player 1 uses a strategy wherein she denies with probability 0.3 and confesses with probability 0.7.

Mixed strategies

		player 2	
		Confess	Deny
Player 1	Confess	(-5, -5)	(0,-10)
	Deny	(-10, 0)	(-2, -2)

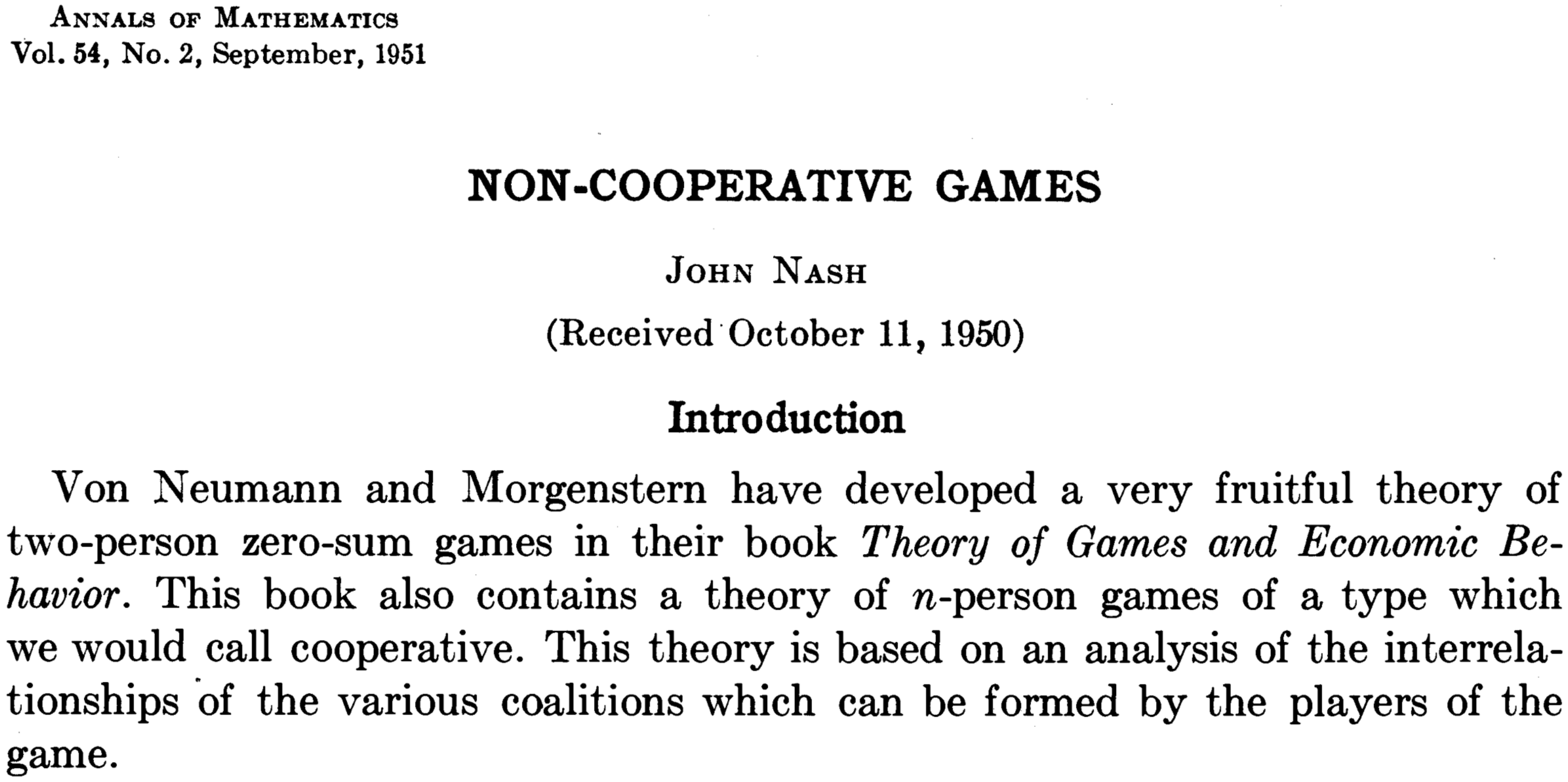
Player 1 uses a strategy wherein she denies with probability 0.3 and confesses with probability 0.7.

Rewards:

Player 1 mixed, player 2 confesses	$0.7 \times (-5) + 0.3 \times (-10) = -6.5$
Player 1 mixed, player 2 denies	$0.7 \times 0 + 0.3 \times (-2) = -0.6$

Mixed strategies

There always exist mixed-strategy Nash equilibria!



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NON-COOPERATIVE GAMES

JOHN NASH

(Received October 11, 1950)

Introduction

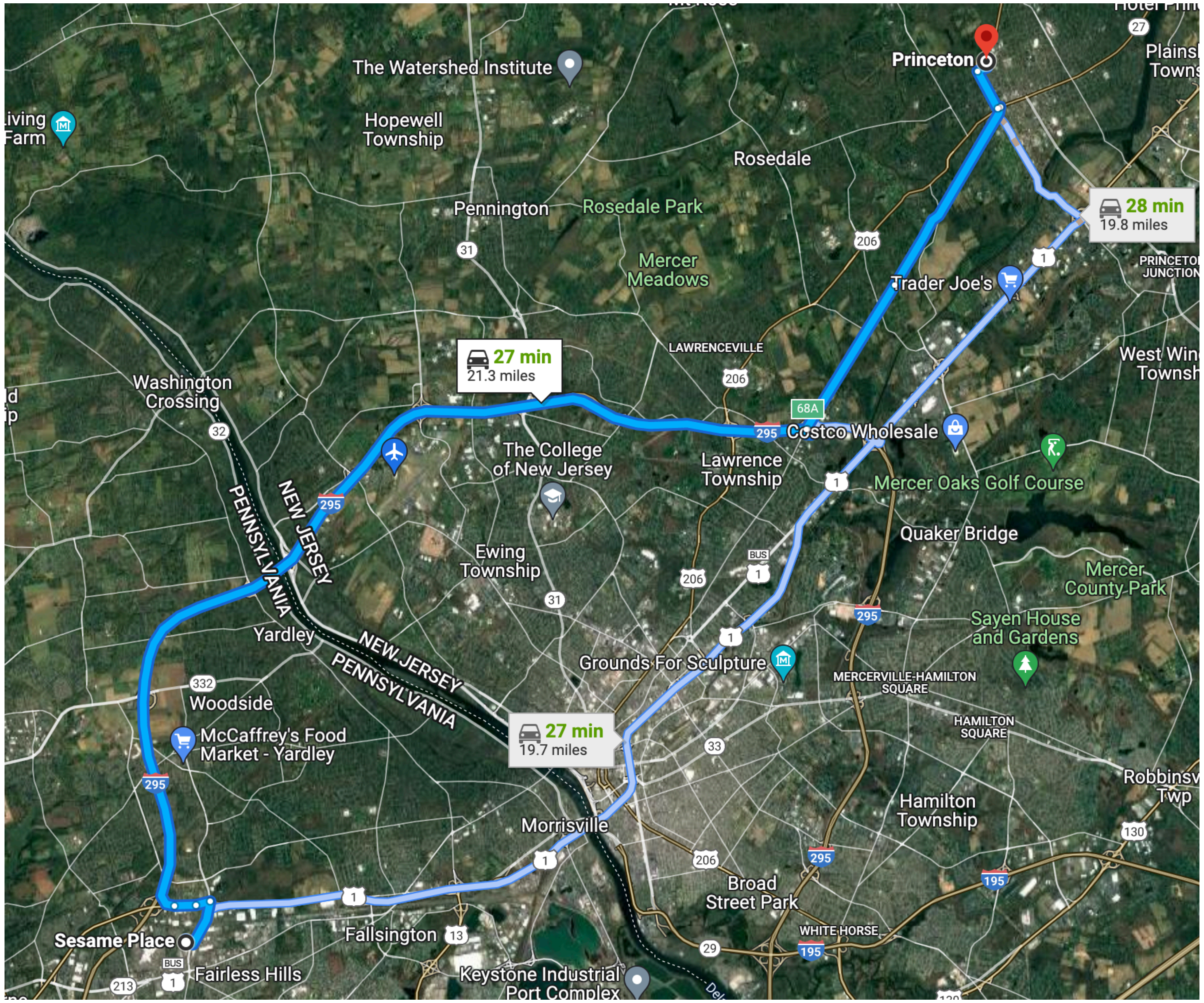
Von Neumann and Morgenstern have developed a very fruitful theory of two-person zero-sum games in their book *Theory of Games and Economic Behavior*. This book also contains a theory of n-person games of a type which we would call cooperative. This theory is based on an analysis of the interrelationships of the various coalitions which can be formed by the players of the game.

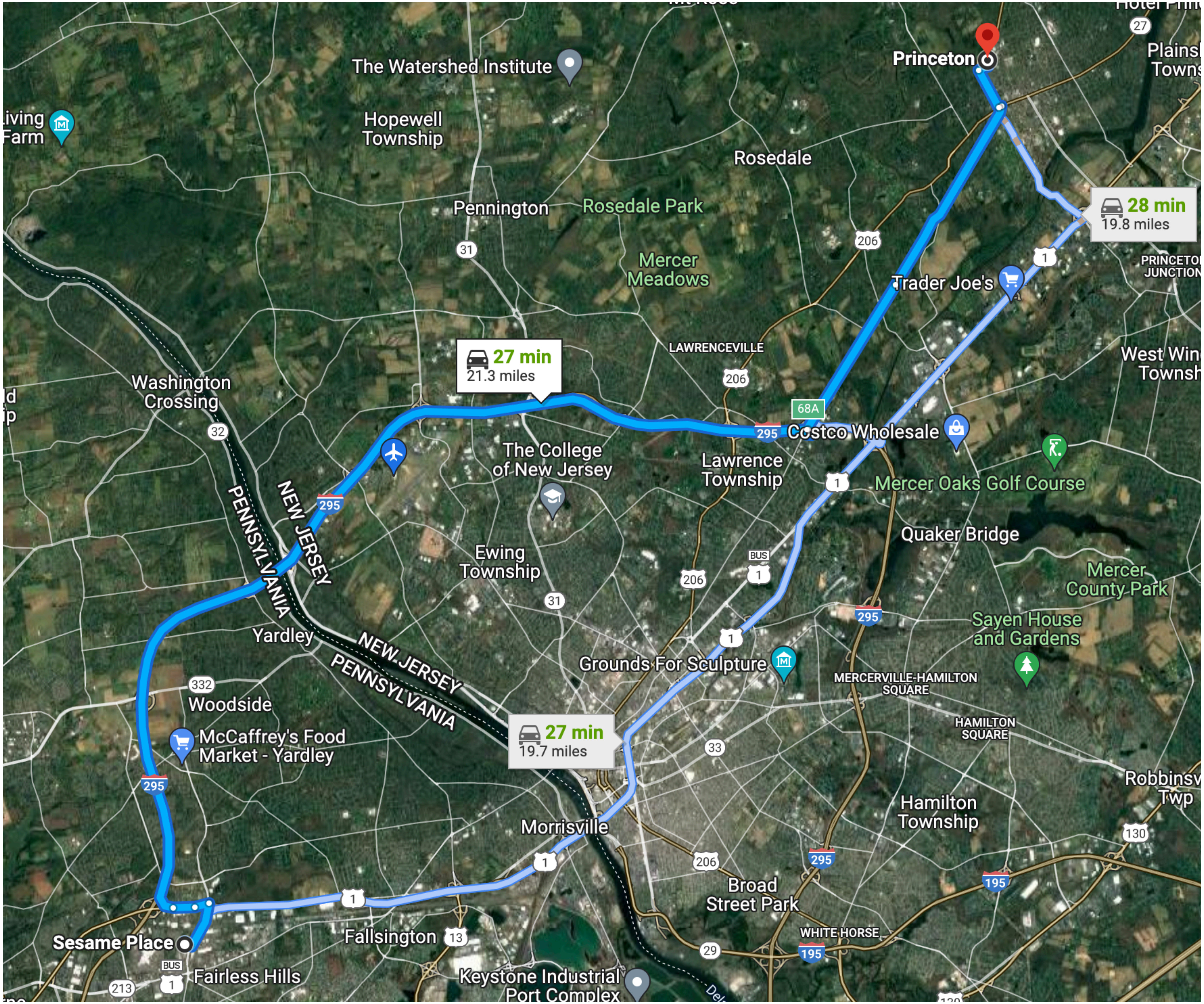
Mixed-strategy Nash equilibrium

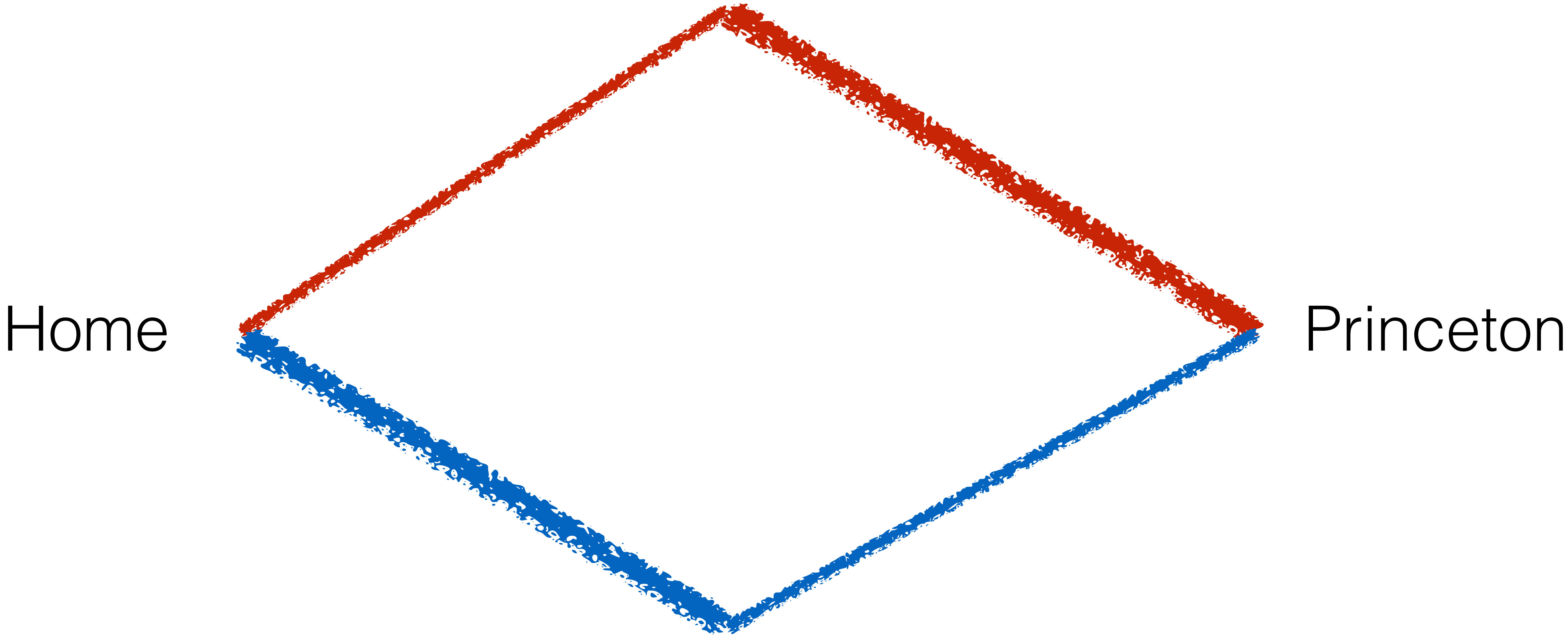
How do mixed-strategy Nash equilibria arise in real life?

One segment of a commute to Princeton:

US-1 or I-295?



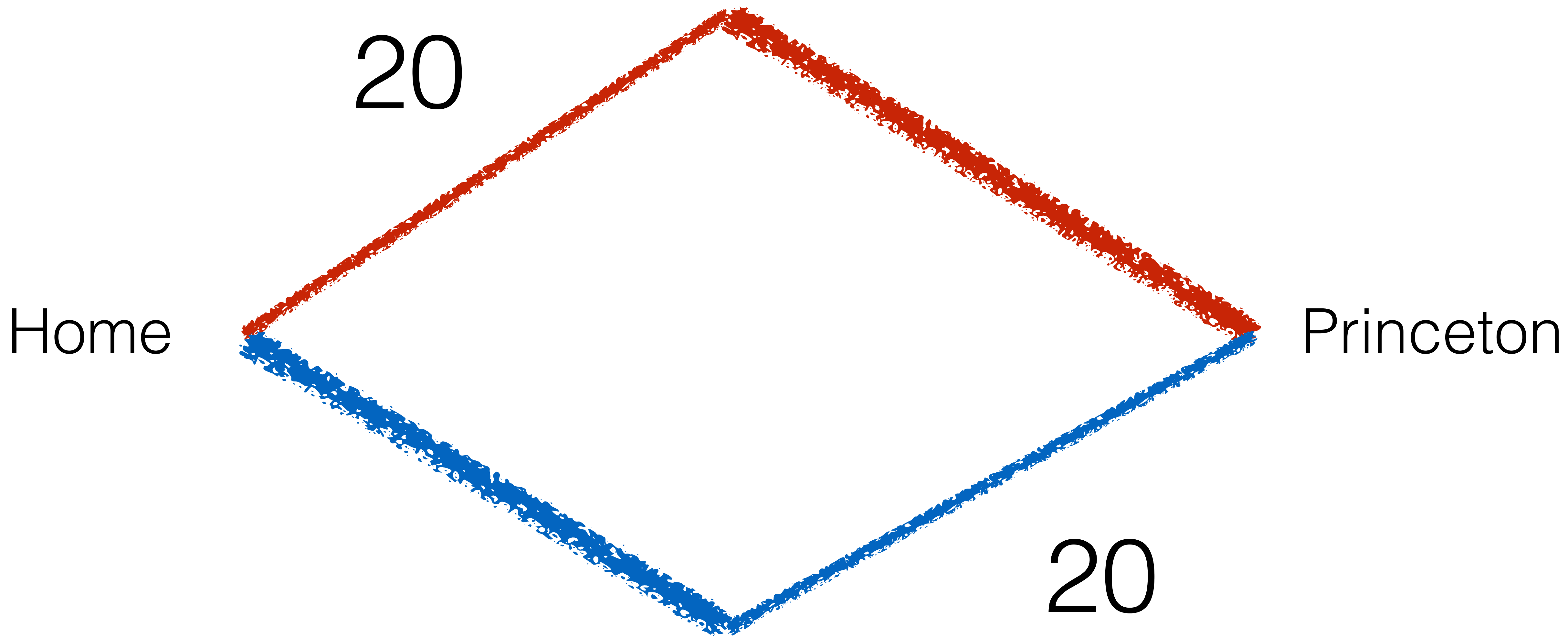

Princeton
28 min
19.8 miles
27 min
21.3 miles
27 min
19.7 miles
Sesame Place
The Watershed Institute
Hopewell Township
Rosedale
Pennington
Rosedale Park
Mercer Meadows
Trader Joe's
PRINCETON JUNCTION
LAWRENCEVILLE
West Windsor Township
Washington Crossing
68A
Costco Wholesale
Lawrence Township
The College of New Jersey
Mercer Oaks Golf Course
Quaker Bridge
Ewing Township
Mercer County Park
PENNSYLVANIA
NEW JERSEY
Yardley
Sayen House and Gardens
Grounds For Sculpture
MERCERVILLE-HAMILTON SQUARE
HAMILTON SQUARE
Woodside
McCaffrey's Food Market - Yardley
Robbinsville Twp
Hamilton Township
Morrisville
Broad Street Park
WHITE HORSE
Fallsington
Fairless Hills
Keystone Industrial Port Complex
295
1
206
31
32
332
33
27
130
195
13
29
213
BUS 1



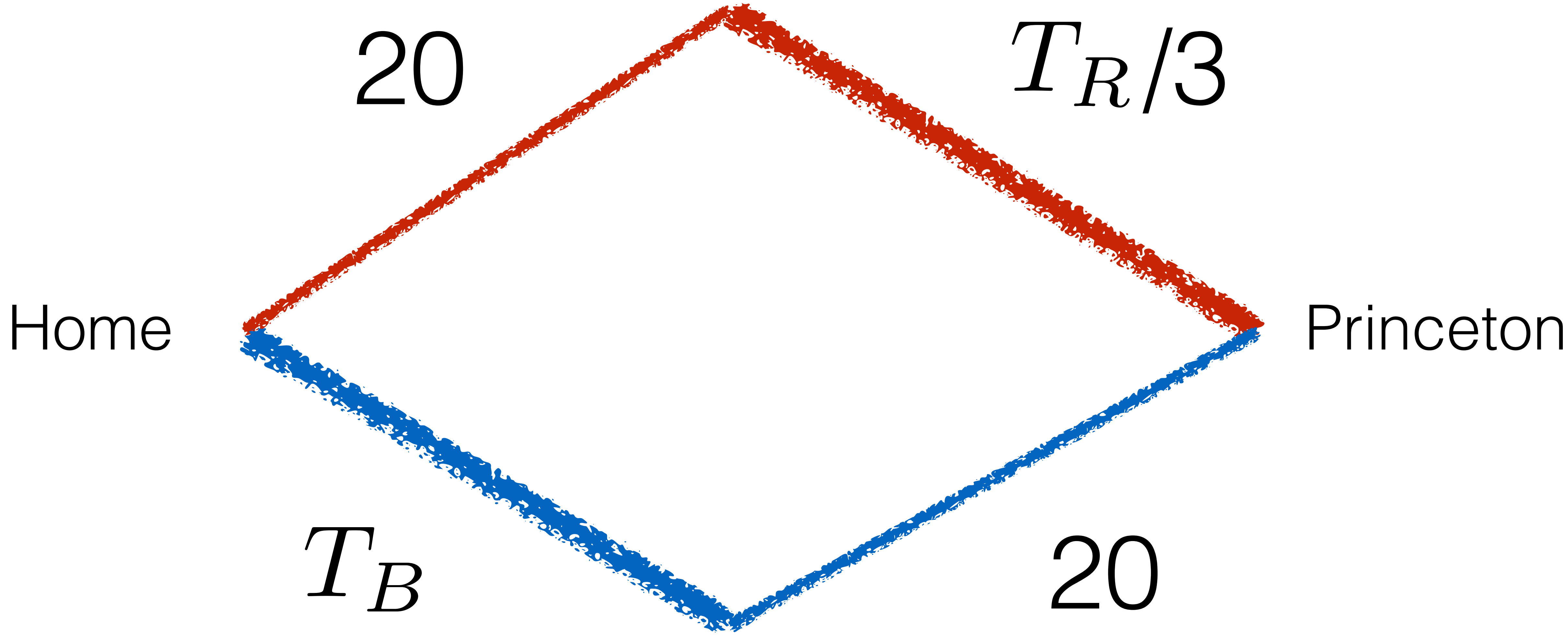
Home

Princeton

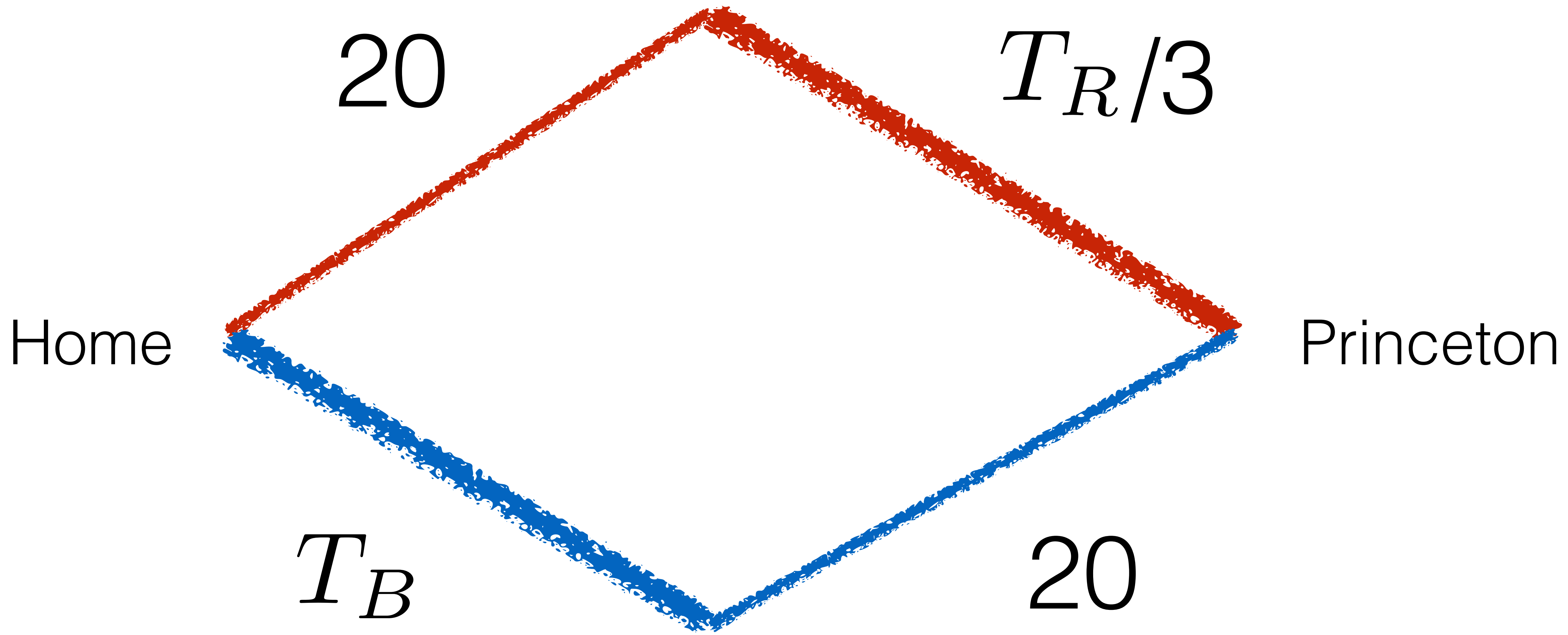
Travel time in minutes



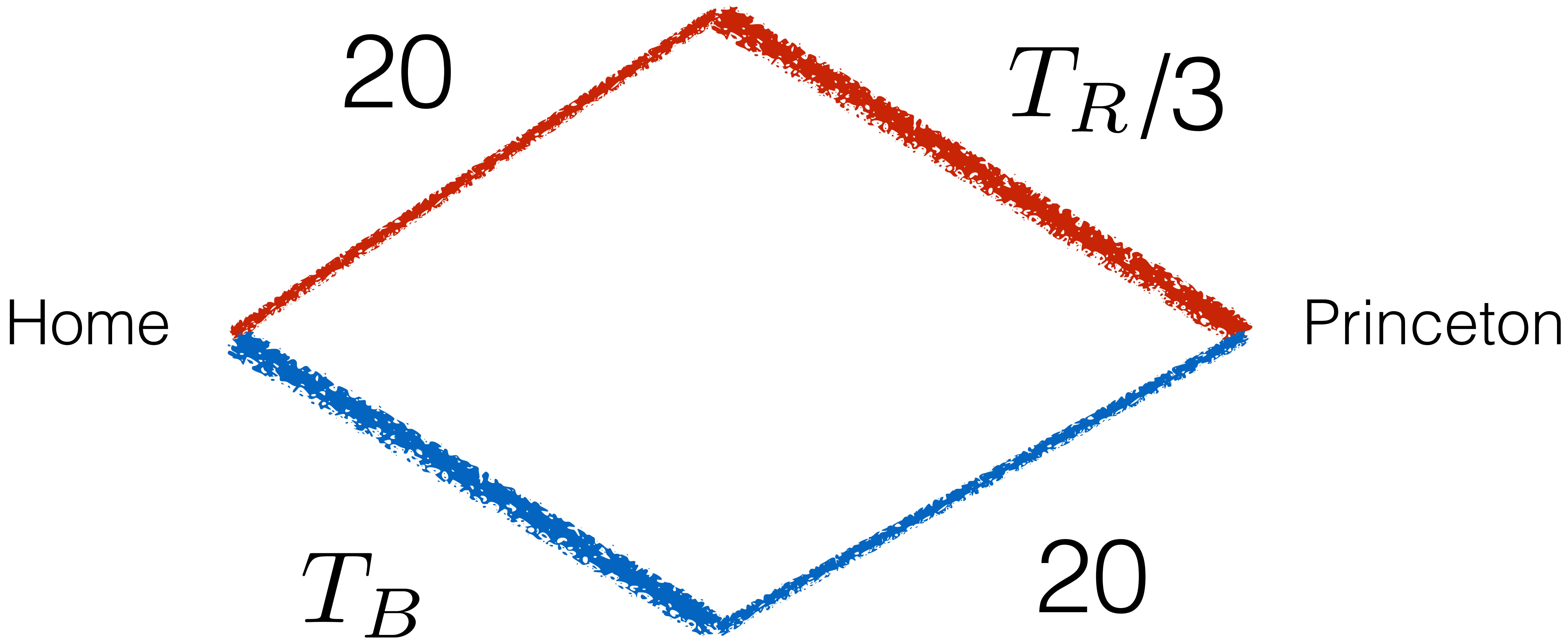
Travel time in minutes



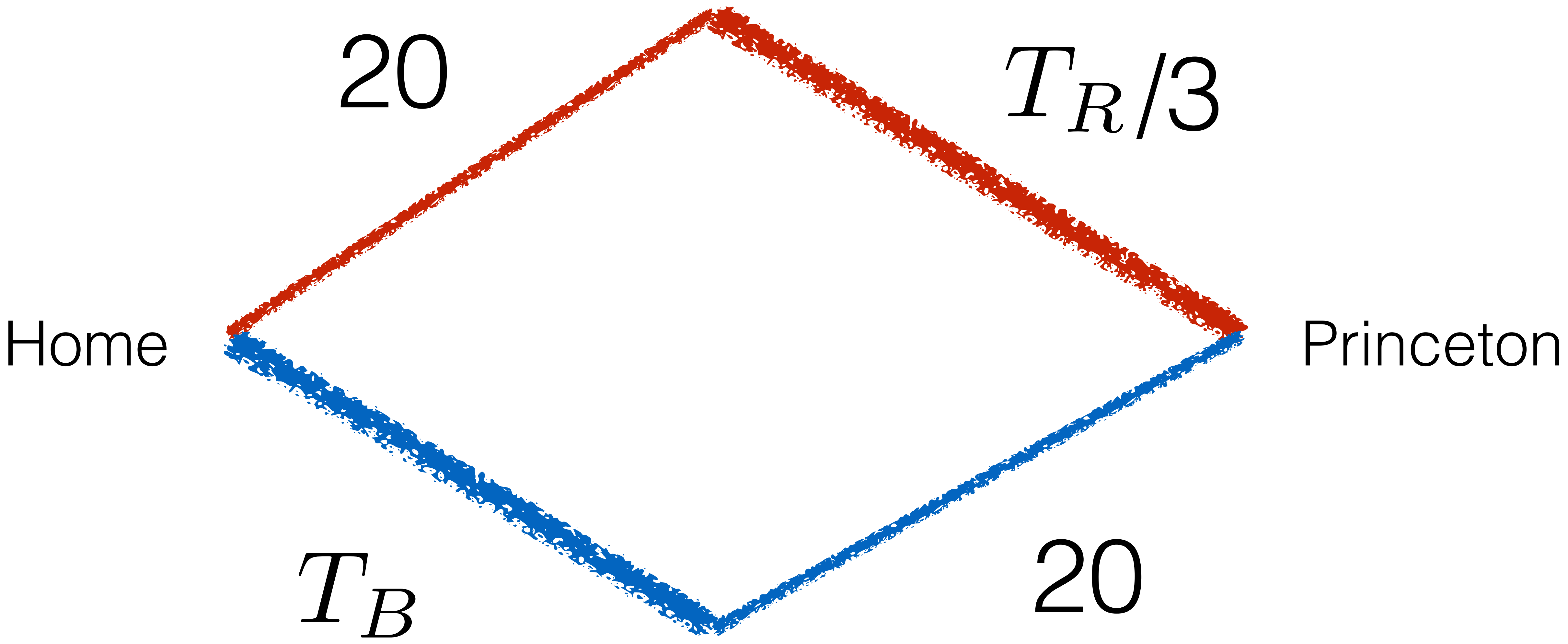
Parts of the routes are congested and the travel time depends on how many cars pick the route. T_B pick blue and T_R pick red



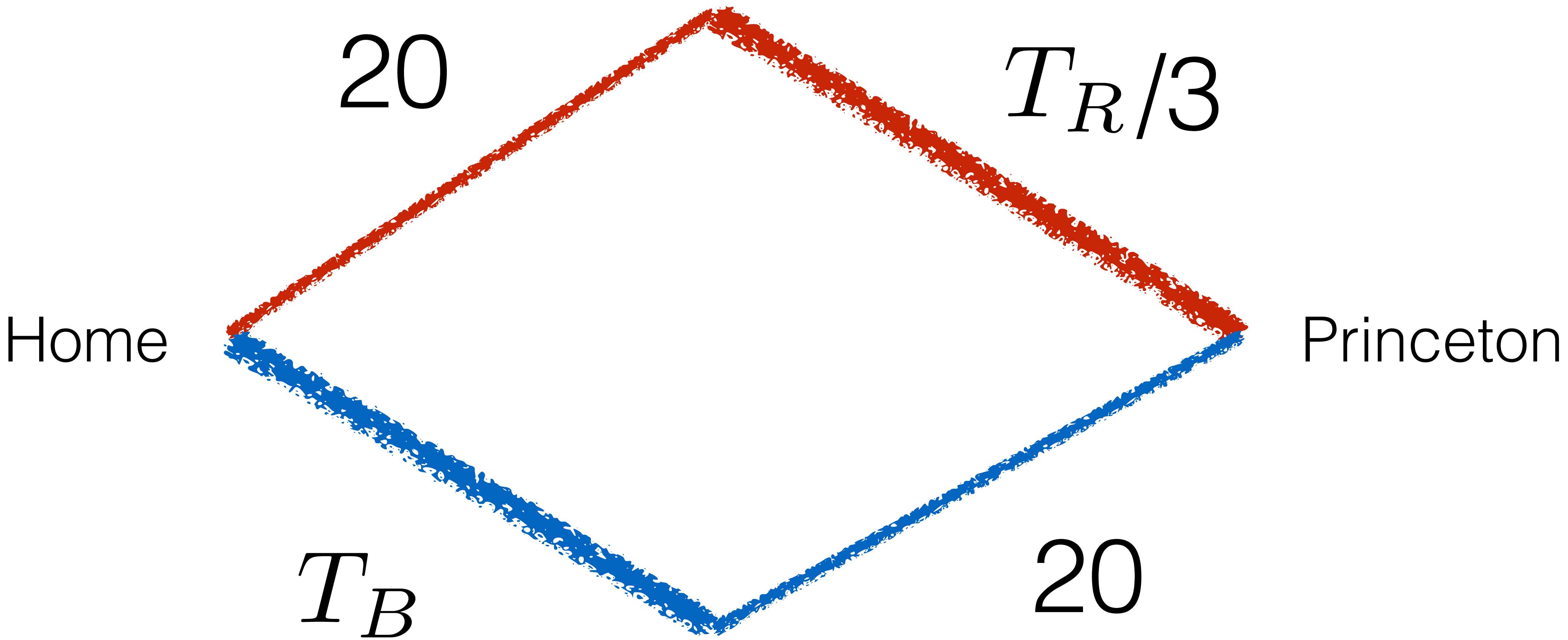
How many cars will choose the red route?



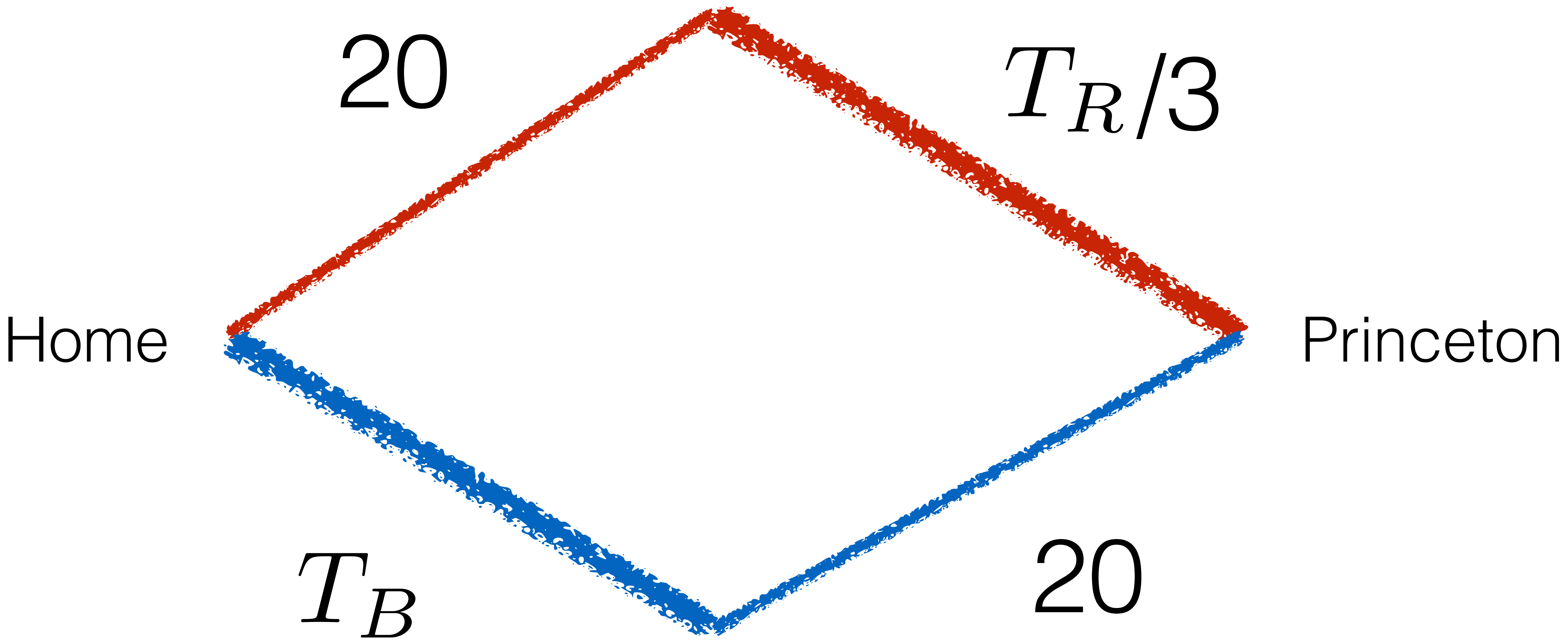
Let’s play a game!



Show of hands, how many pick blue and how many pick red?

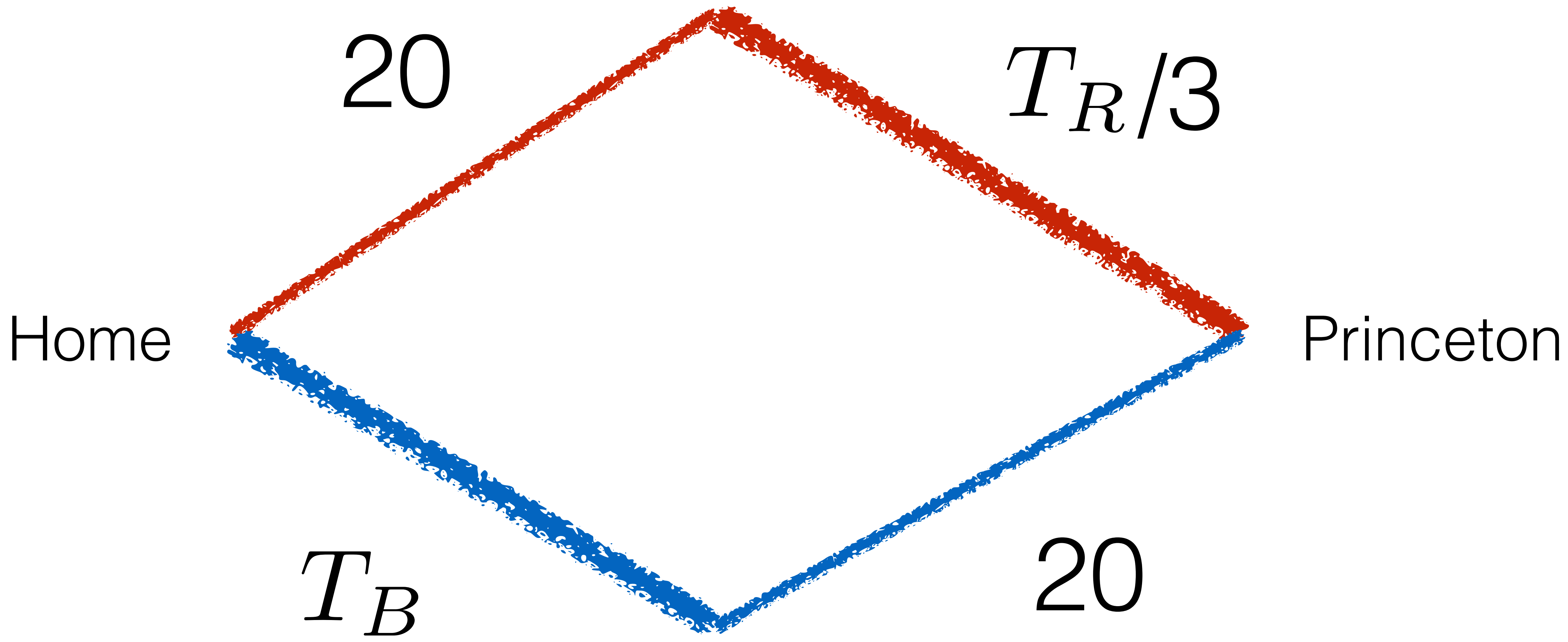


Do you want to change your route?



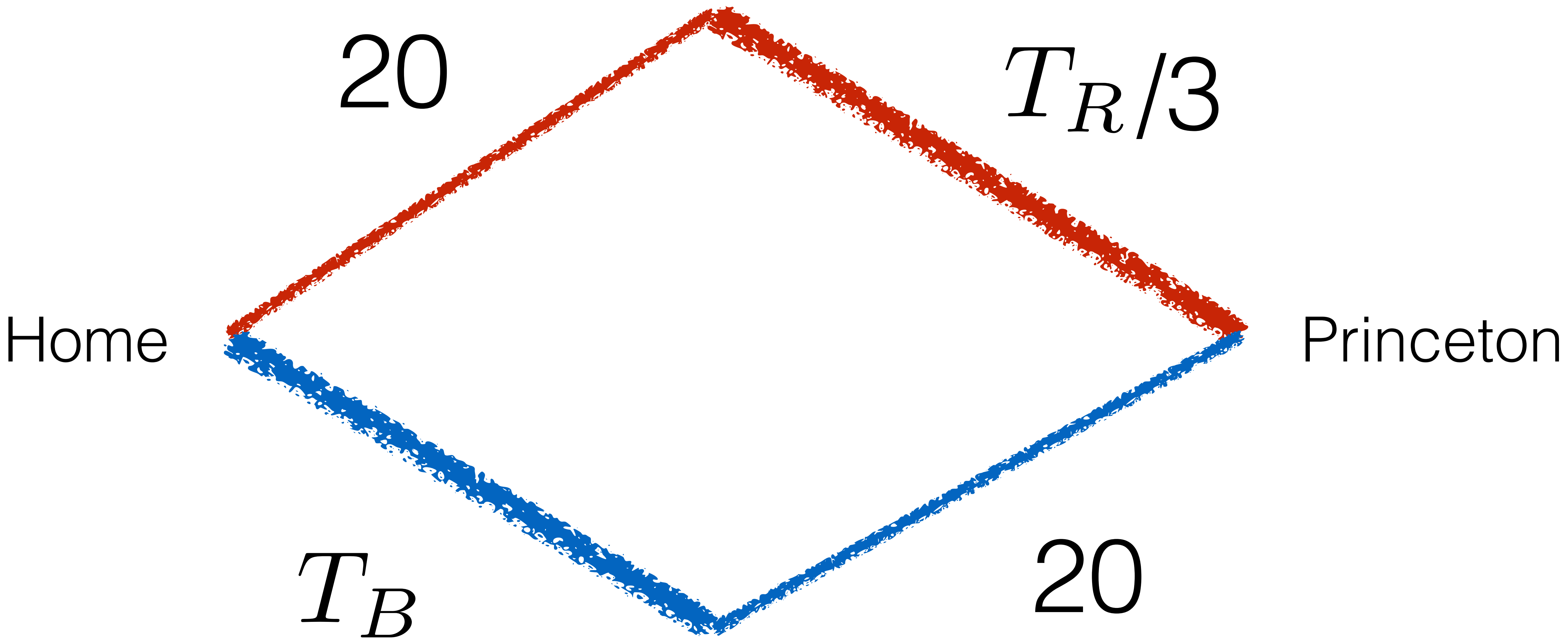
If 20 + $T_R/3$ < T_B + 20

What happens?



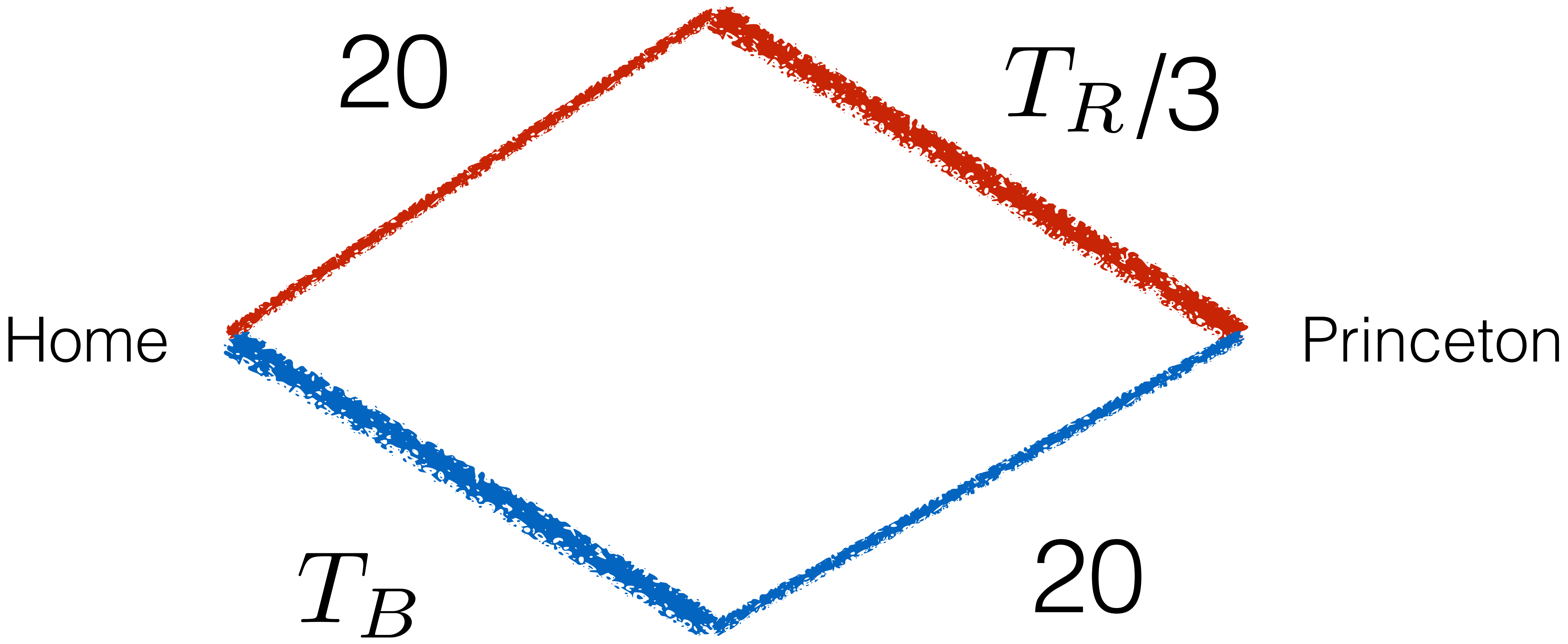
If $20 + T_R/3 < T_B + 20$

T_R should increase until the travel times are equal



But if $20 + T_R/3 > T_B + 20$

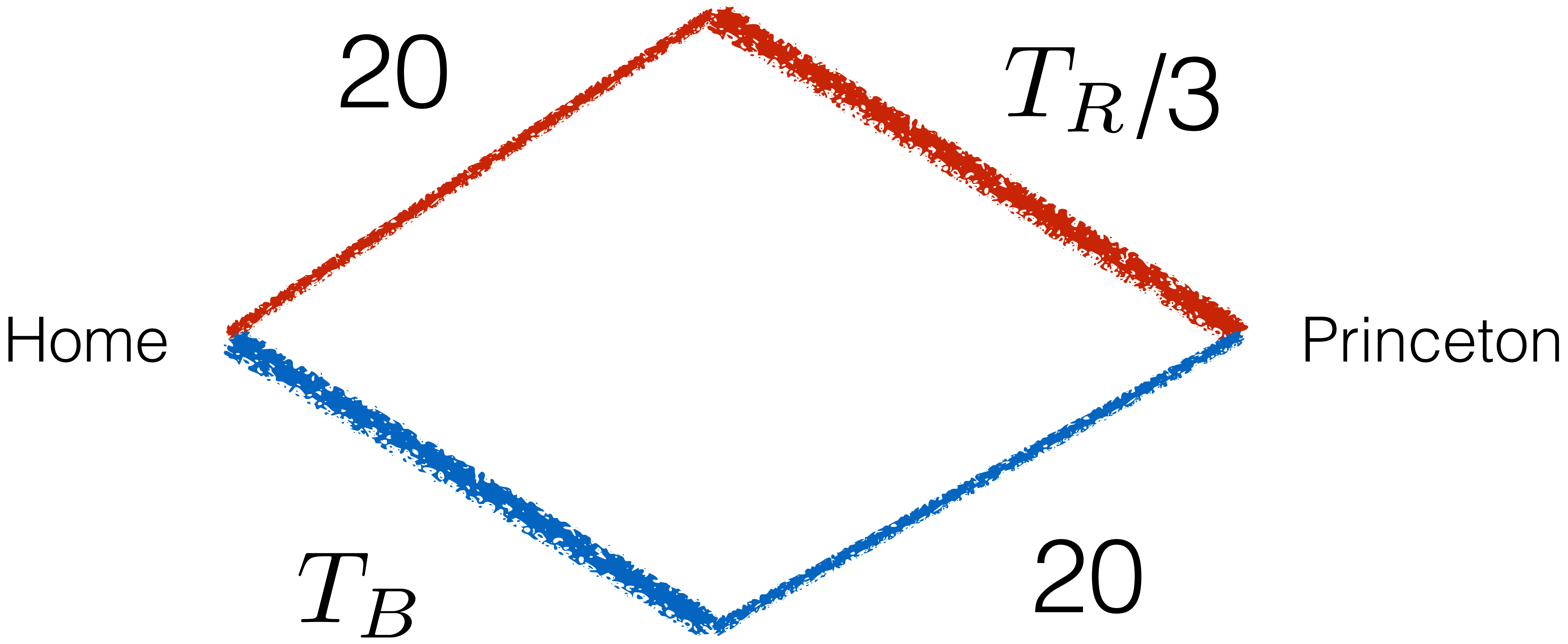
T_R should decrease until the travel times are equal



At equilibrium,

$$20 + T_R/3 = T_B + 20$$

$$T_R = 3\,T_B$$

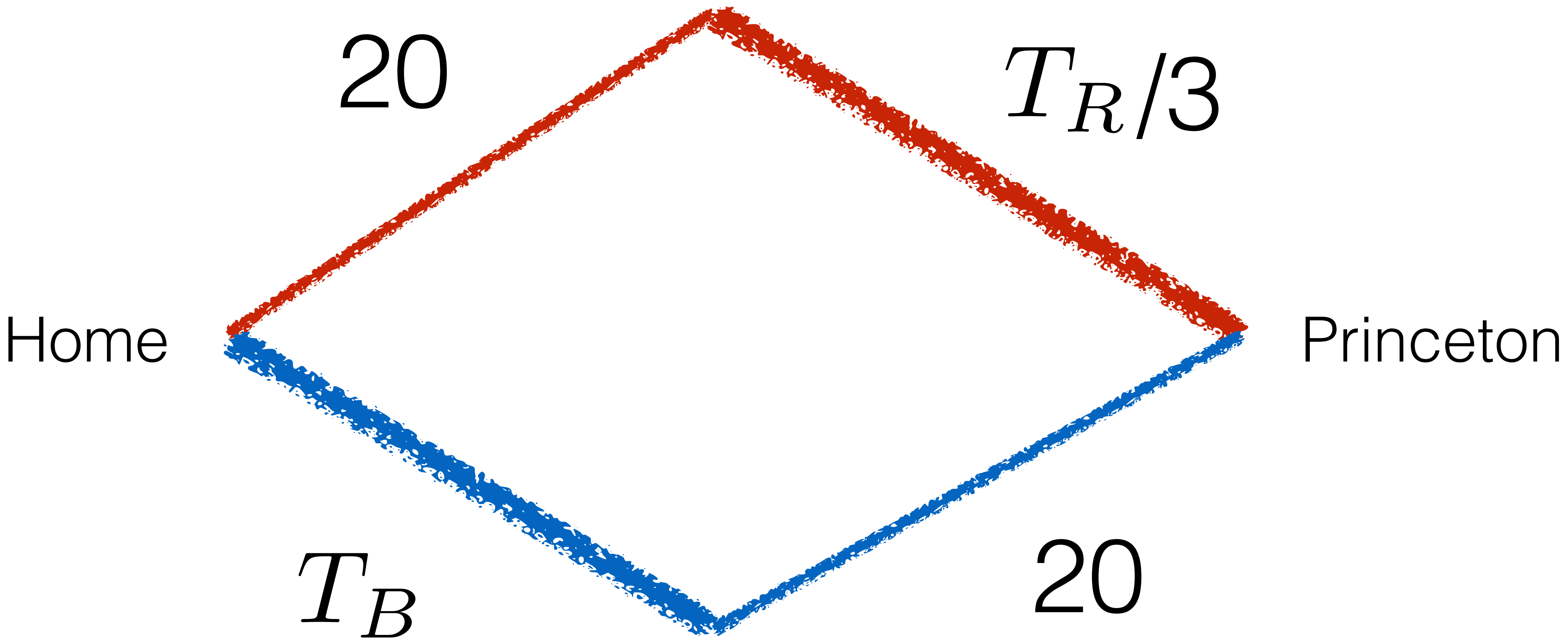


At equilibrium,

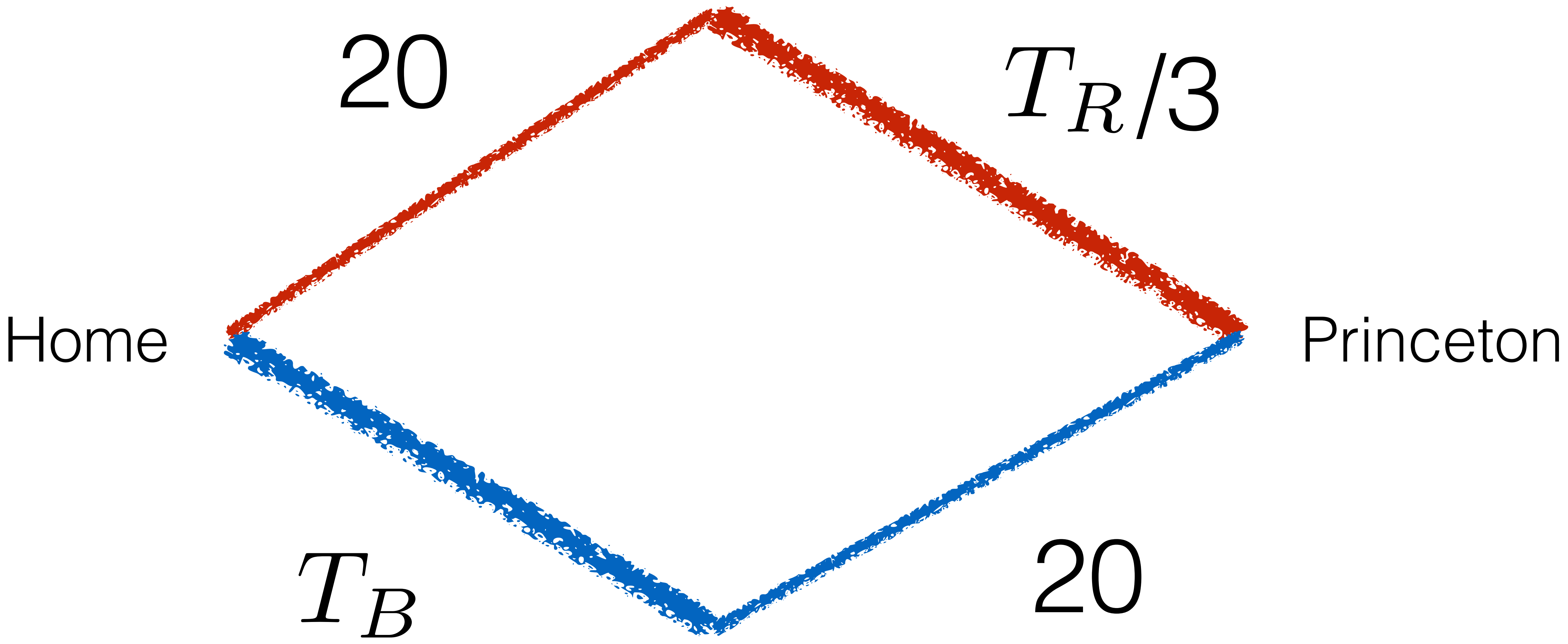
$$20 + T_R/3 = T_B + 20$$

$$T_R = 3\,T_B$$

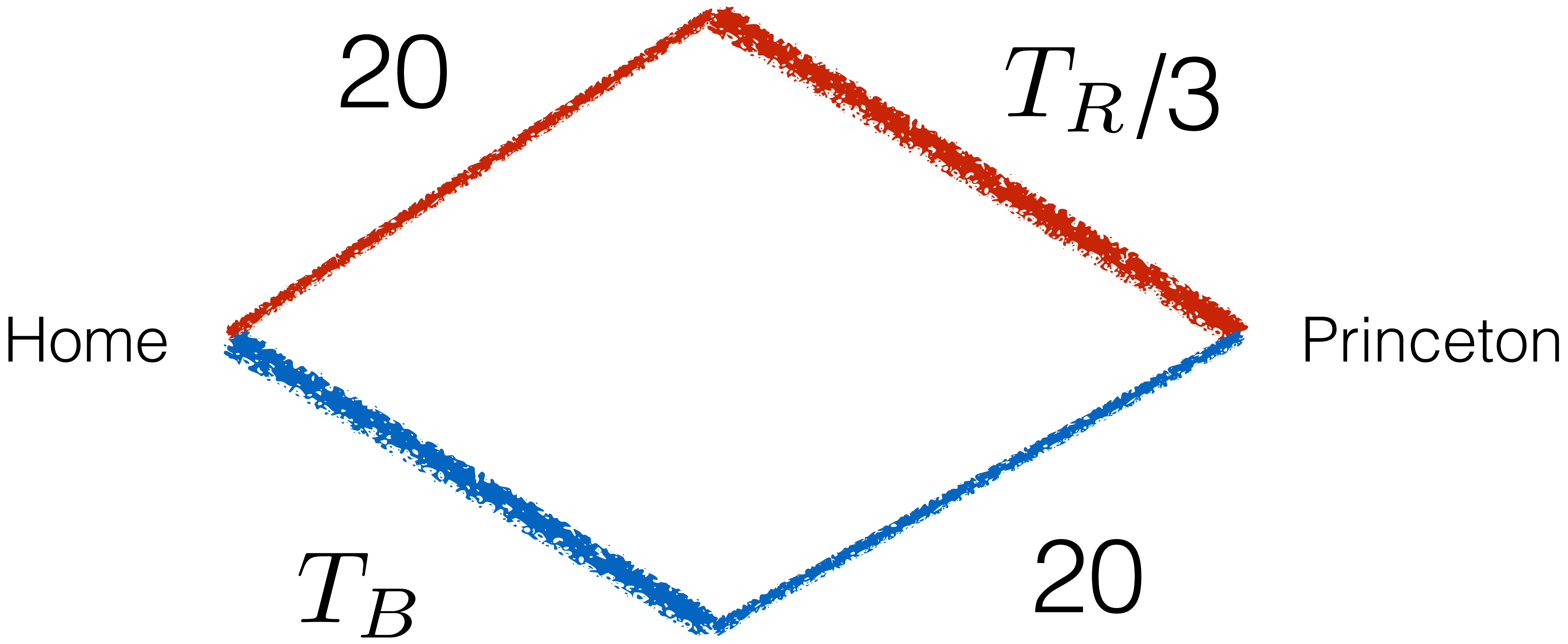
i.e., 3/4 of the drivers choose the red route.



How can we ensure that 3/4 of drivers choose the red route (since each driver is allowed to make their own decision)?

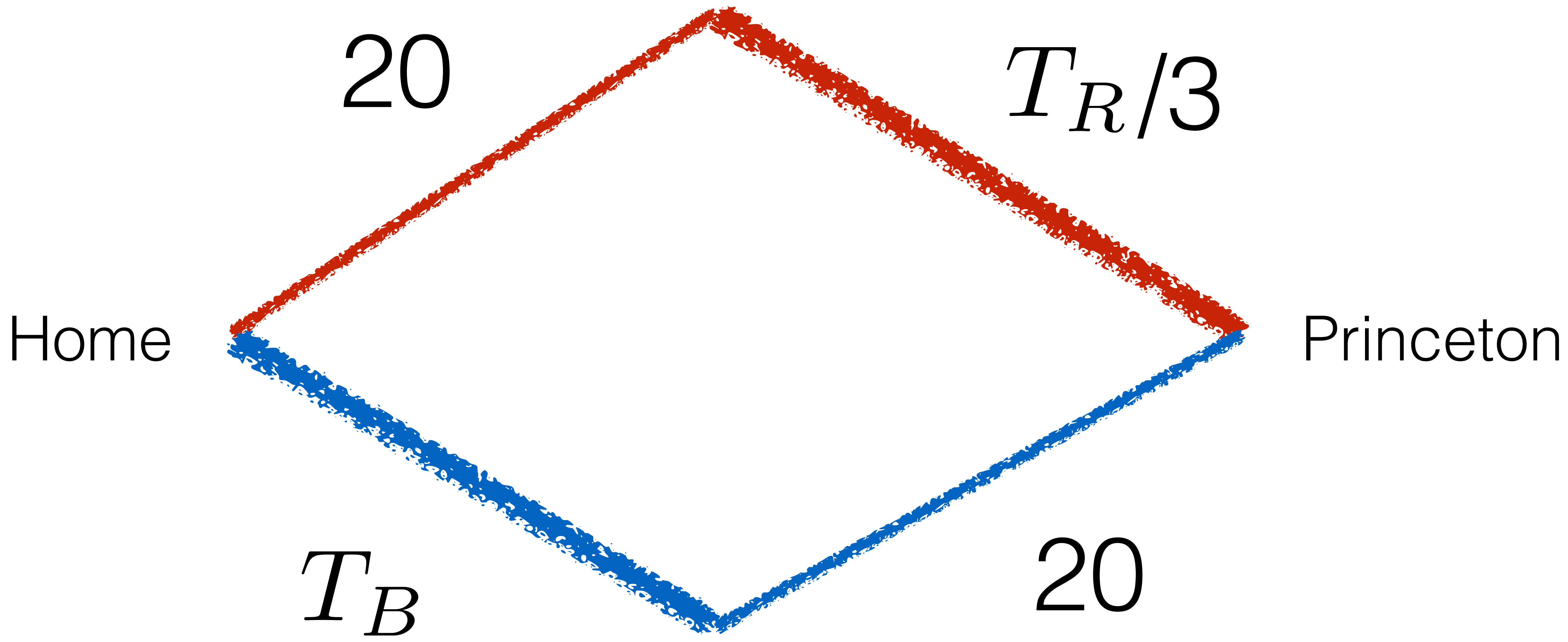


If you commute between these two points every day, choose the red route with prob. 3/4.

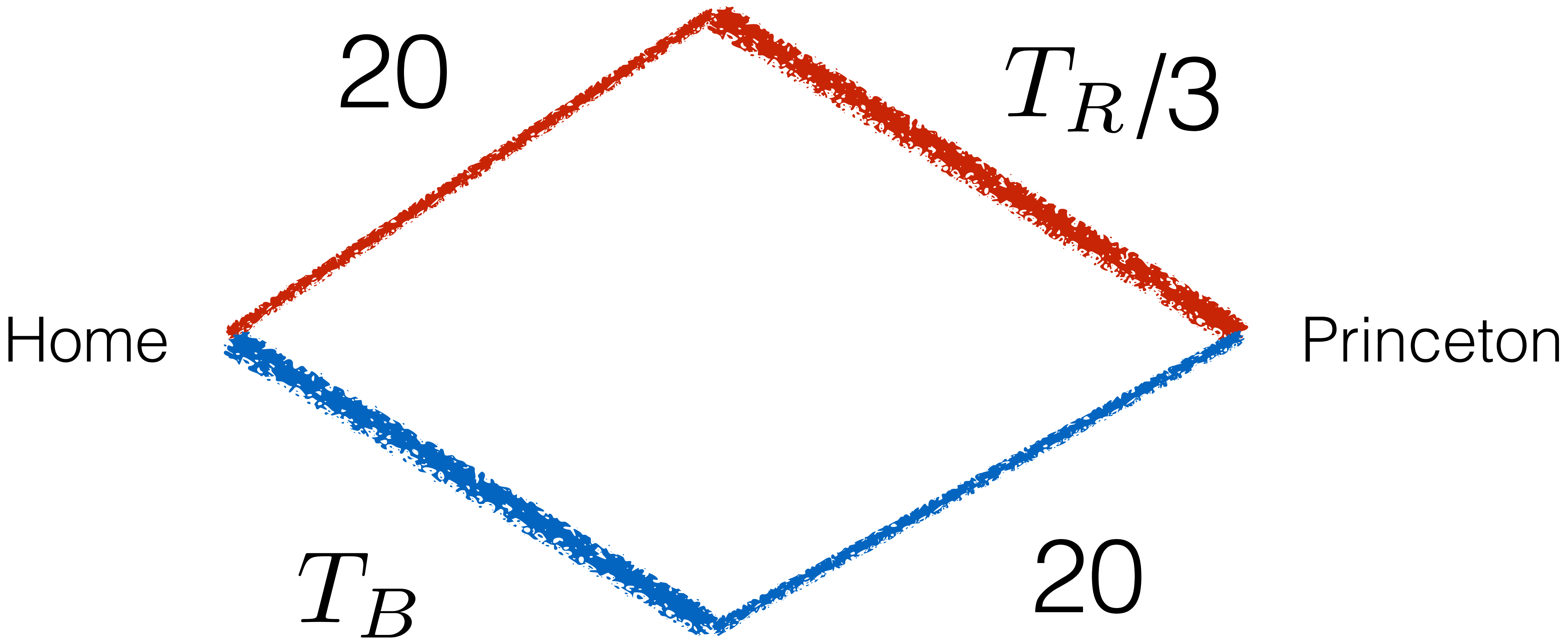


If you commute between these two points every day, then with prob. 3/4, choose the red route.

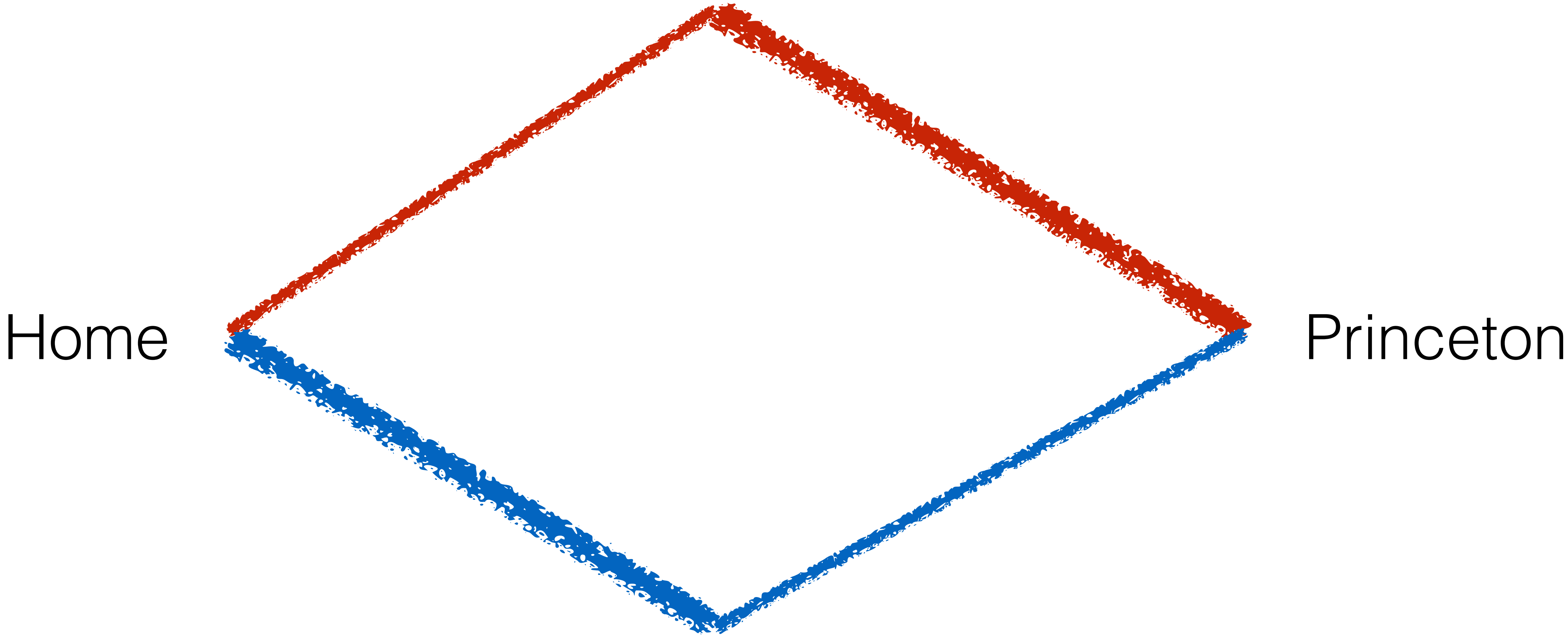
You are playing with a **mixed strategy**!



Every driver is playing with a mixed strategy with probabilities 0.75 and 0.25 for the two pure strategies!



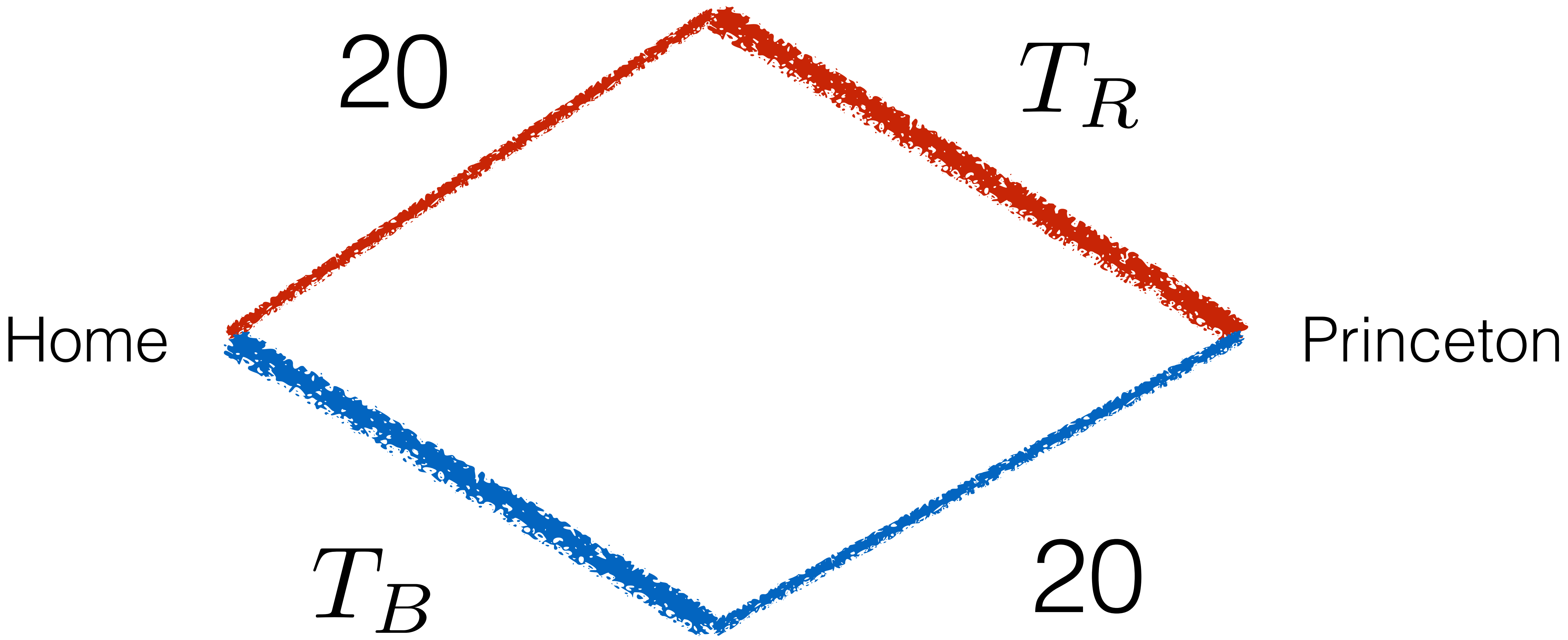
The two probabilities 0.75 and 0.25 for the two pure strategies is a mixed-strategy Nash equilibrium because no-one wants to be the only one to change their route!

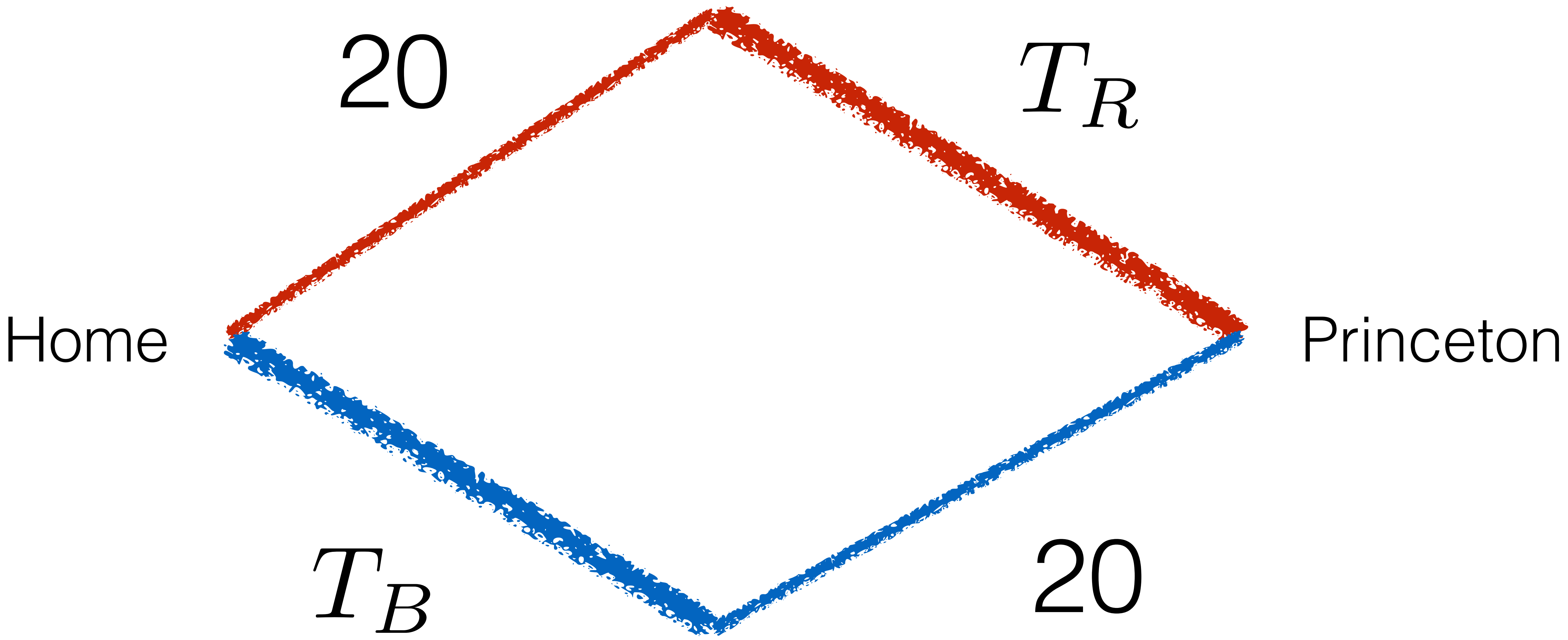


Home

Princeton

Next, we change the game slightly

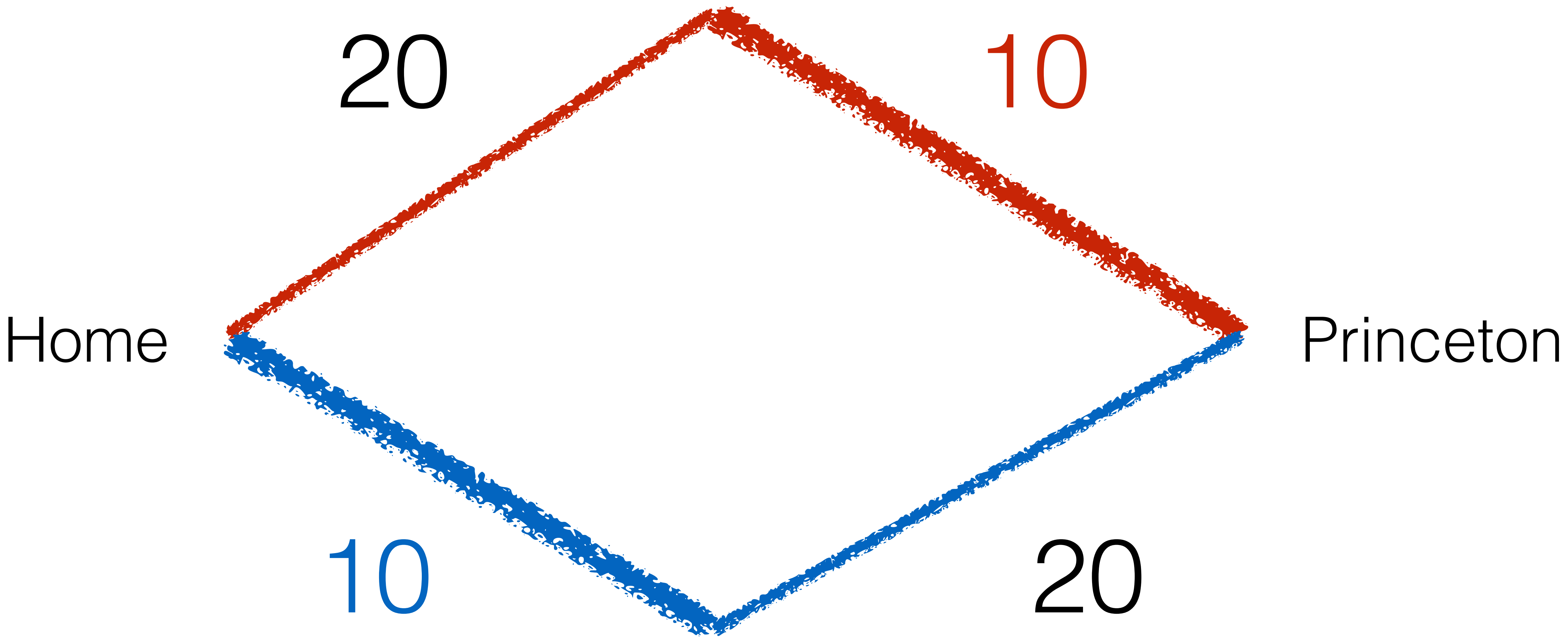

20
T_R
Home
Princeton
T_B
20



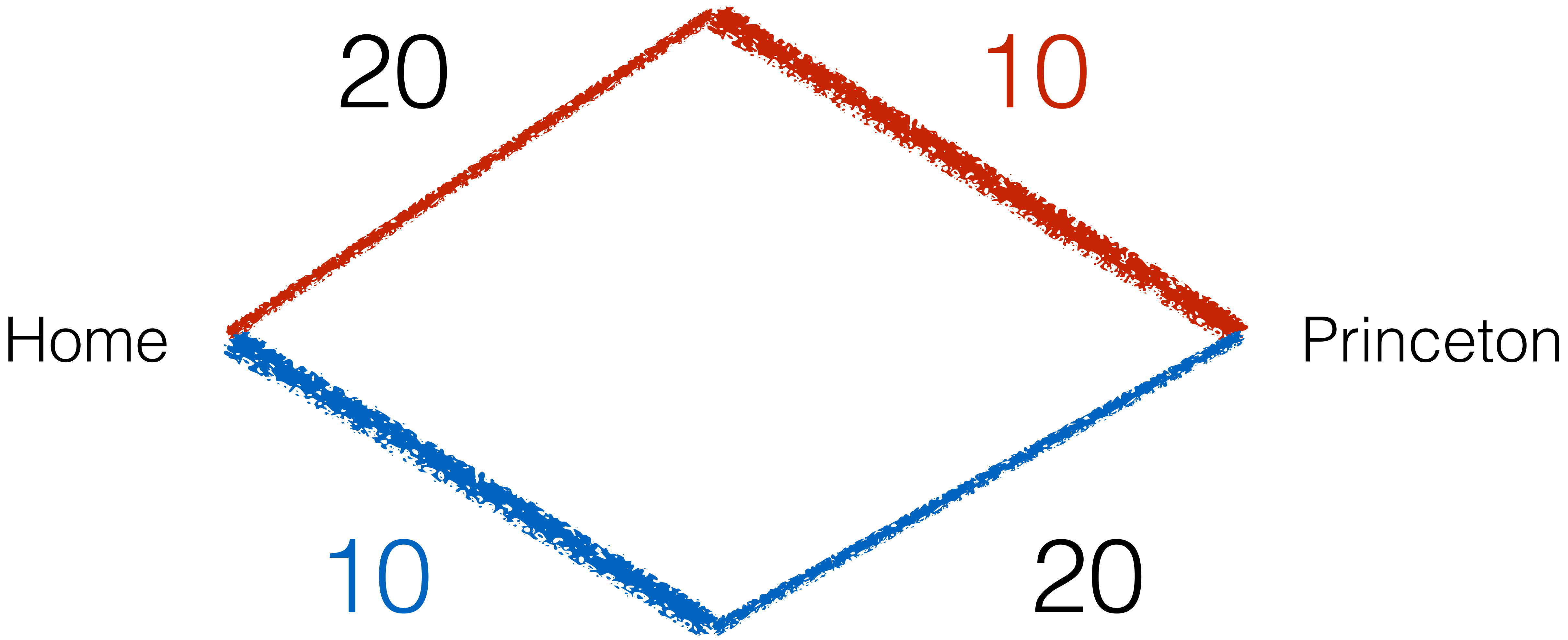
At the equilibrium…

$$20 + T_R = T_B + 20$$

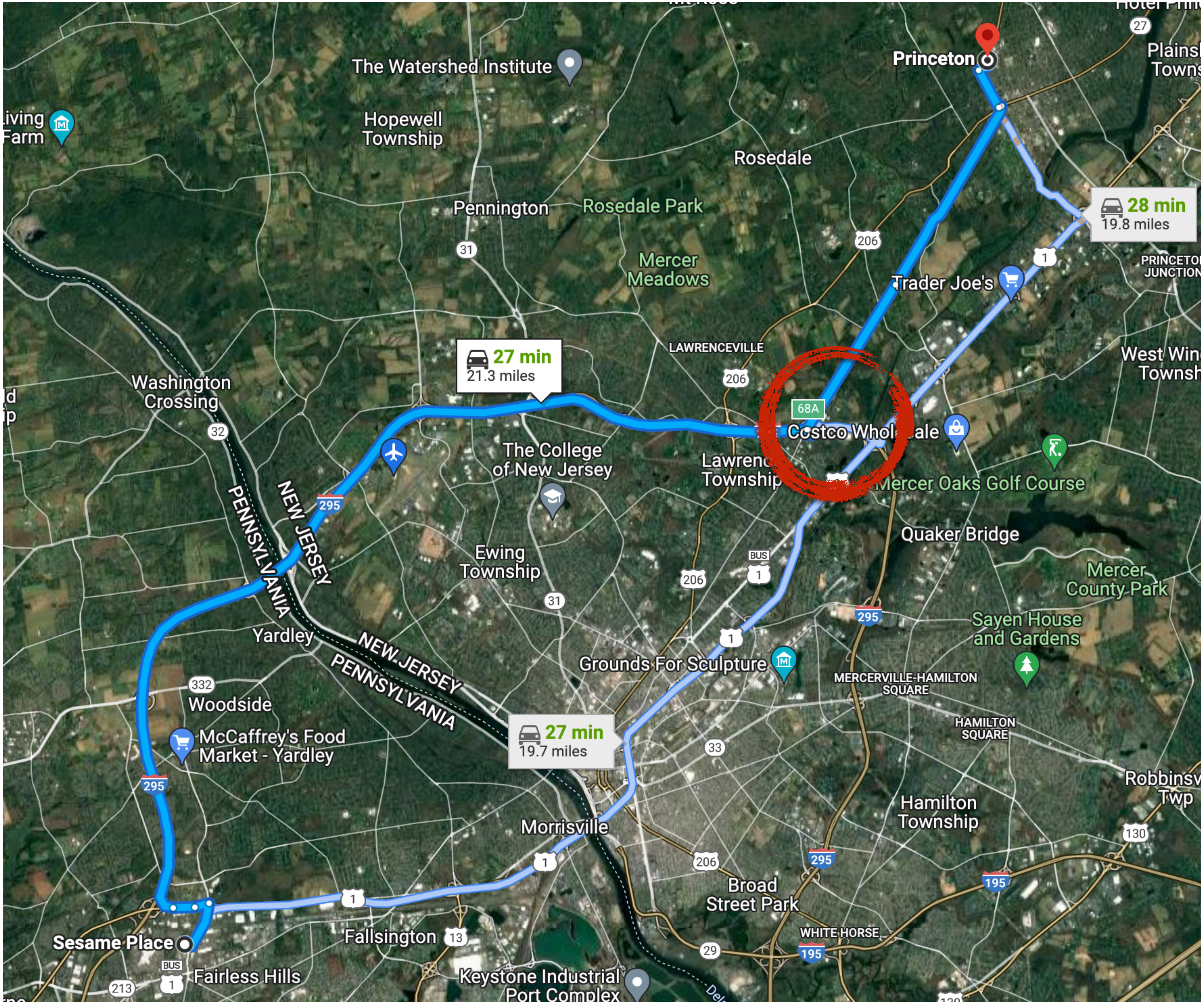
$$T_R = T_B$$

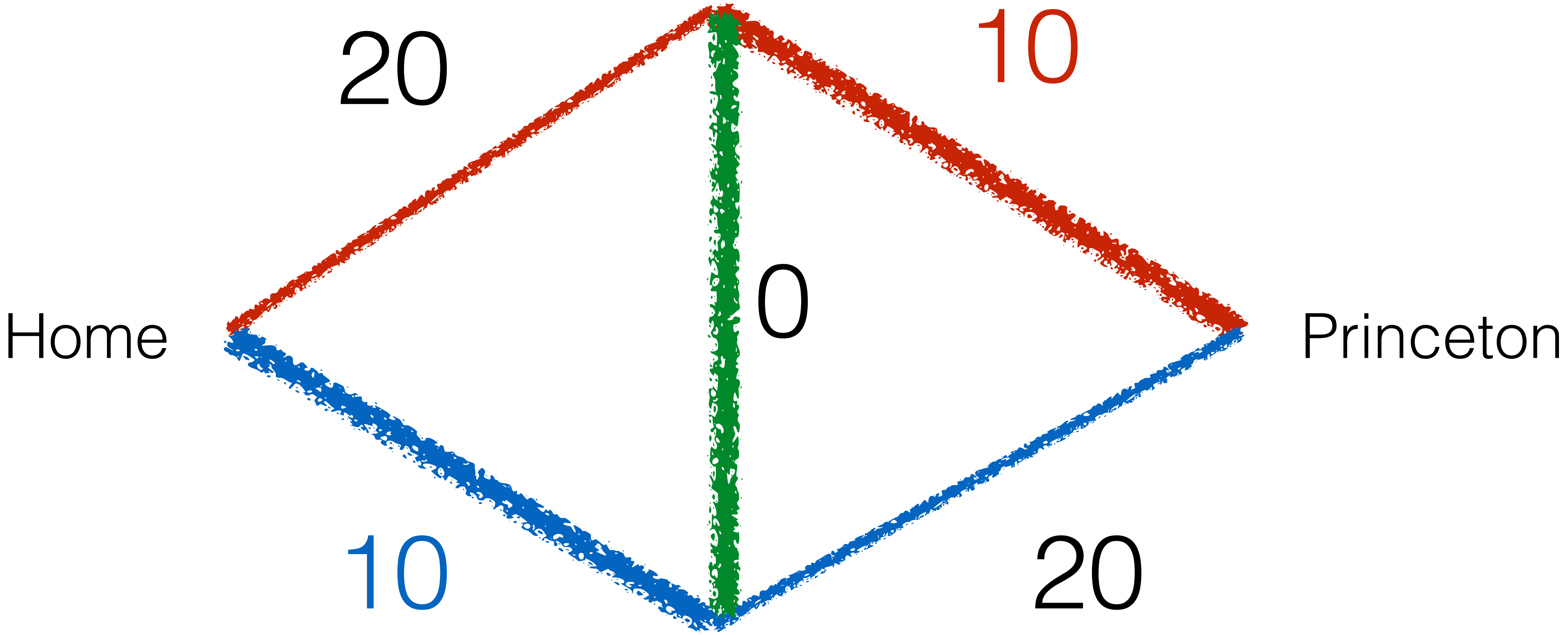


If there are 20 cars in total, this means that $T_R = 10$ and $T_B = 10$.

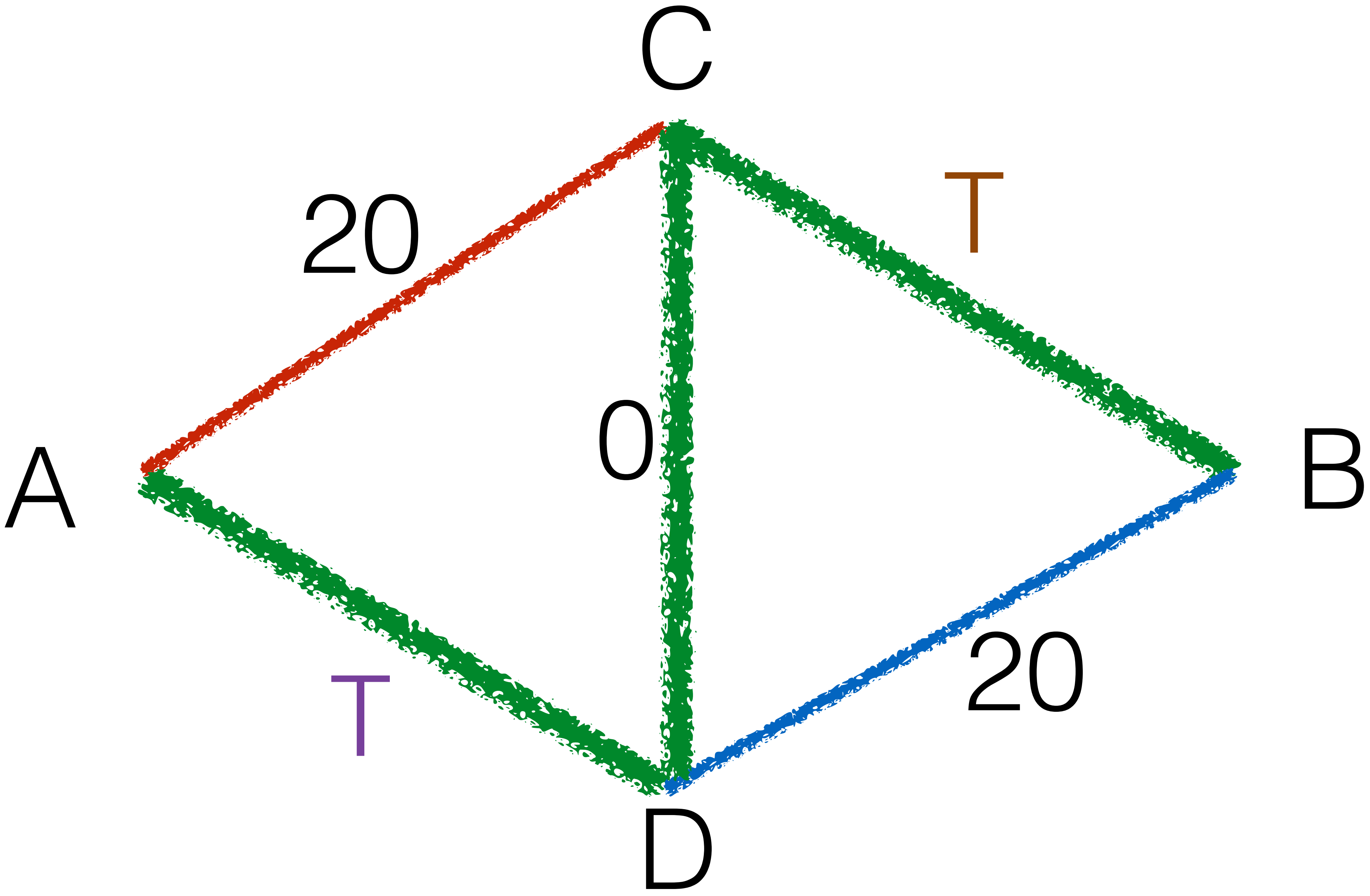


Now, for both routes,
the travel time is **30** min


Princeton
27
Hotel Prin
Plainsb
Towns
The Watershed Institute
Living
Farm
Hopewell
Township
Rosedale
Pennington
Rosedale Park
28 min
19.8 miles
31
206
1
PRINCETON
JUNCTION
Mercer
Meadows
Trader Joe's
LAWRENCEVILLE
27 min
21.3 miles
West Win
Townsh
Washington
Crossing
206
68A
Costco Wholesale
32
Lawrence
Township
Mercer Oaks Golf Course
The College
of New Jersey
295
NEW JERSEY
PENNSYLVANIA
Quaker Bridge
Ewing
Township
BUS
1
206
Mercer
County Park
31
295
Sayen House
and Gardens
Yardley
NEW JERSEY
PENNSYLVANIA
1
Grounds For Sculpture
MERCERVILLE-HAMILTON
SQUARE
332
HAMILTON
SQUARE
Woodside
27 min
19.7 miles
McCaffrey's Food
Market - Yardley
33
Robbinsv
Twp
295
Hamilton
Township
Morrisville
130
1
206
295
195
1
Broad
Street Park
Sesame Place
Fallsington
13
WHITE HORSE
29
195
BUS
1
Fairless Hills
213
Keystone Industrial
Port Complex

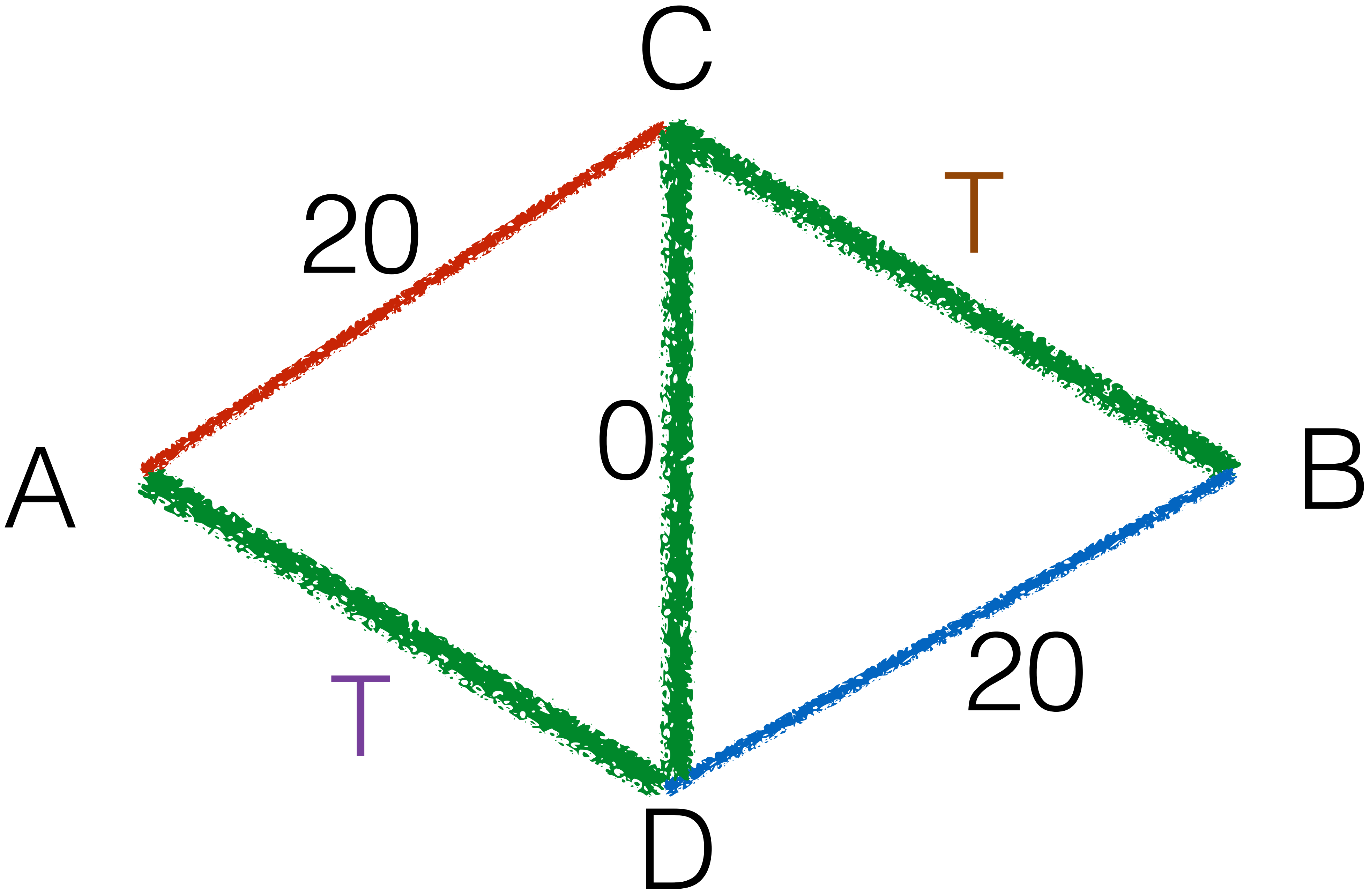


Discussion: what will happen?



Discussion: what will happen?

4 routes: A-C-B; A-D-B, A-C-D-B, A-D-C-B

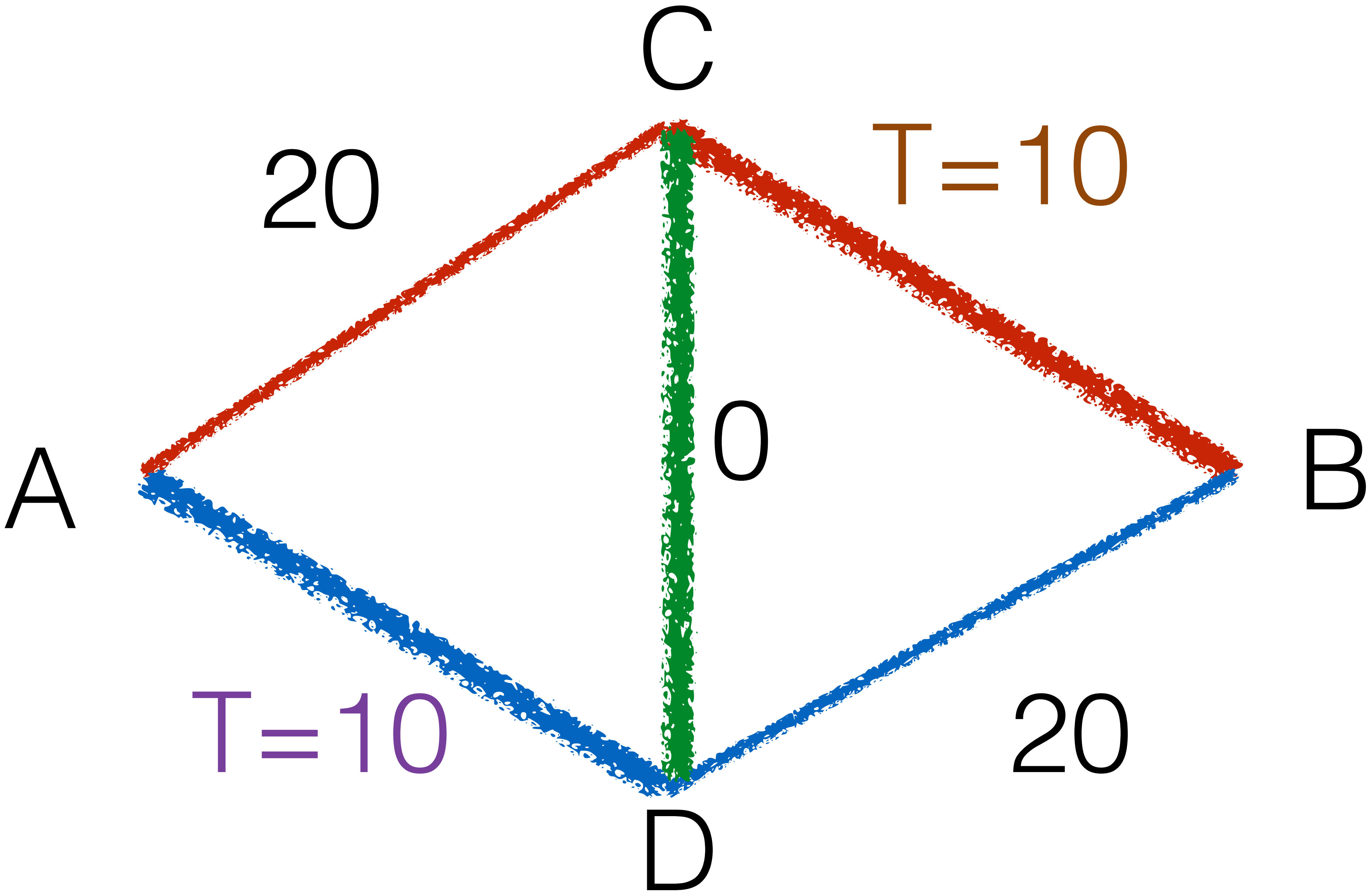


T is the sum of people choosing A-D-B and A-D-C-B

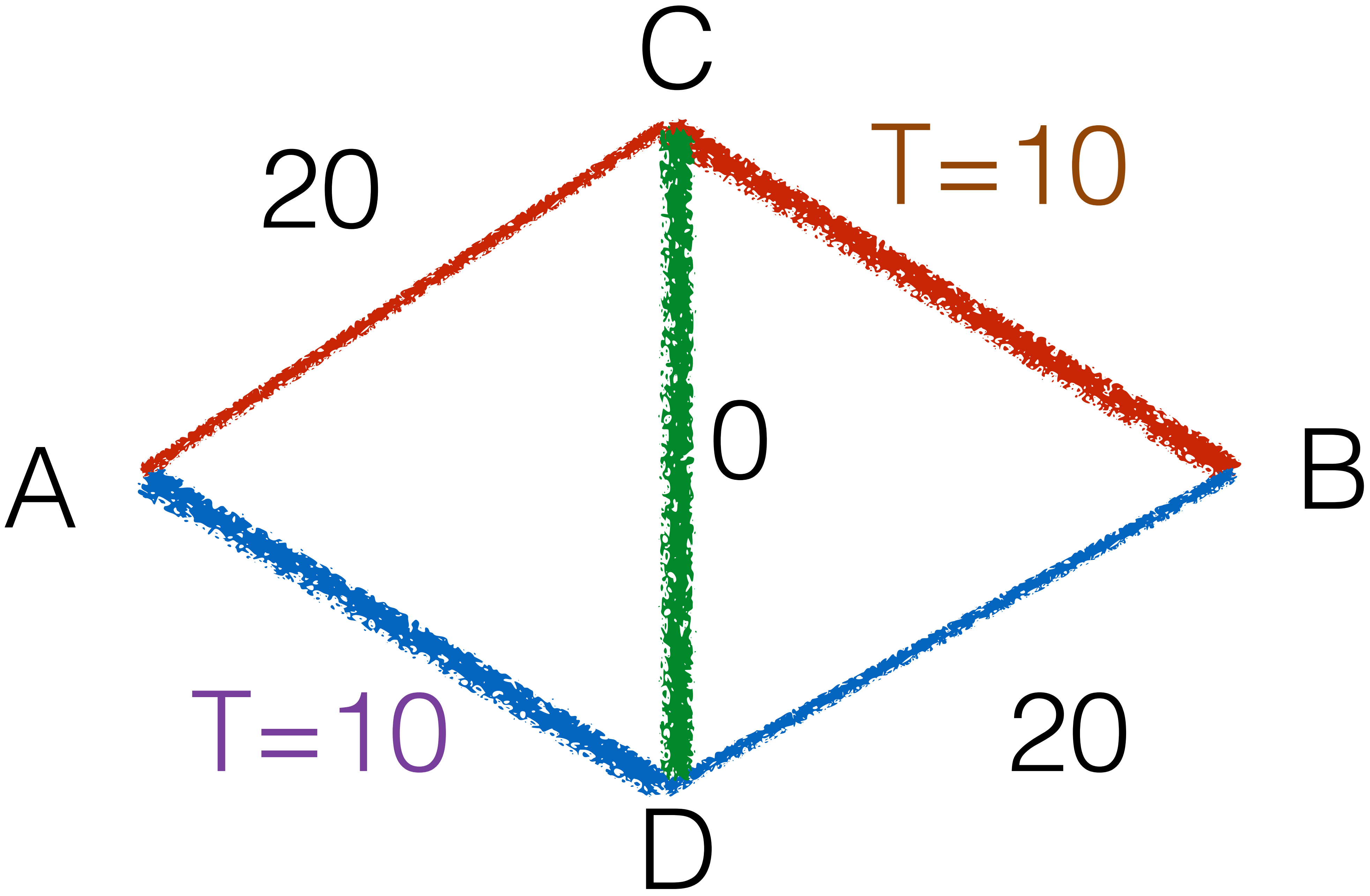
T is the sum of people choosing A-C-B and A-D-C-B

Pick a route:

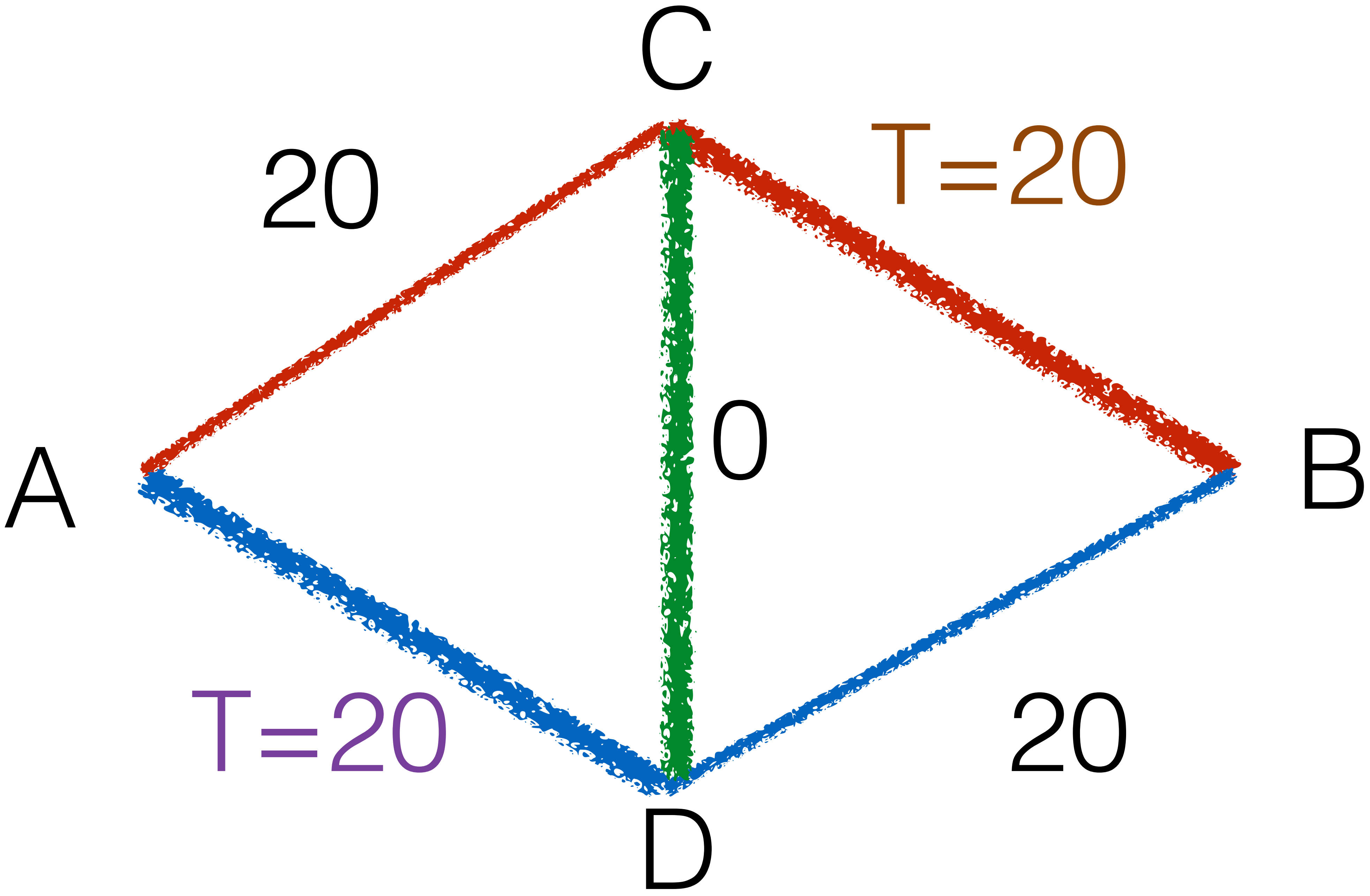
A-C-B; A-D-B, A-C-D-B, A-D-C-B



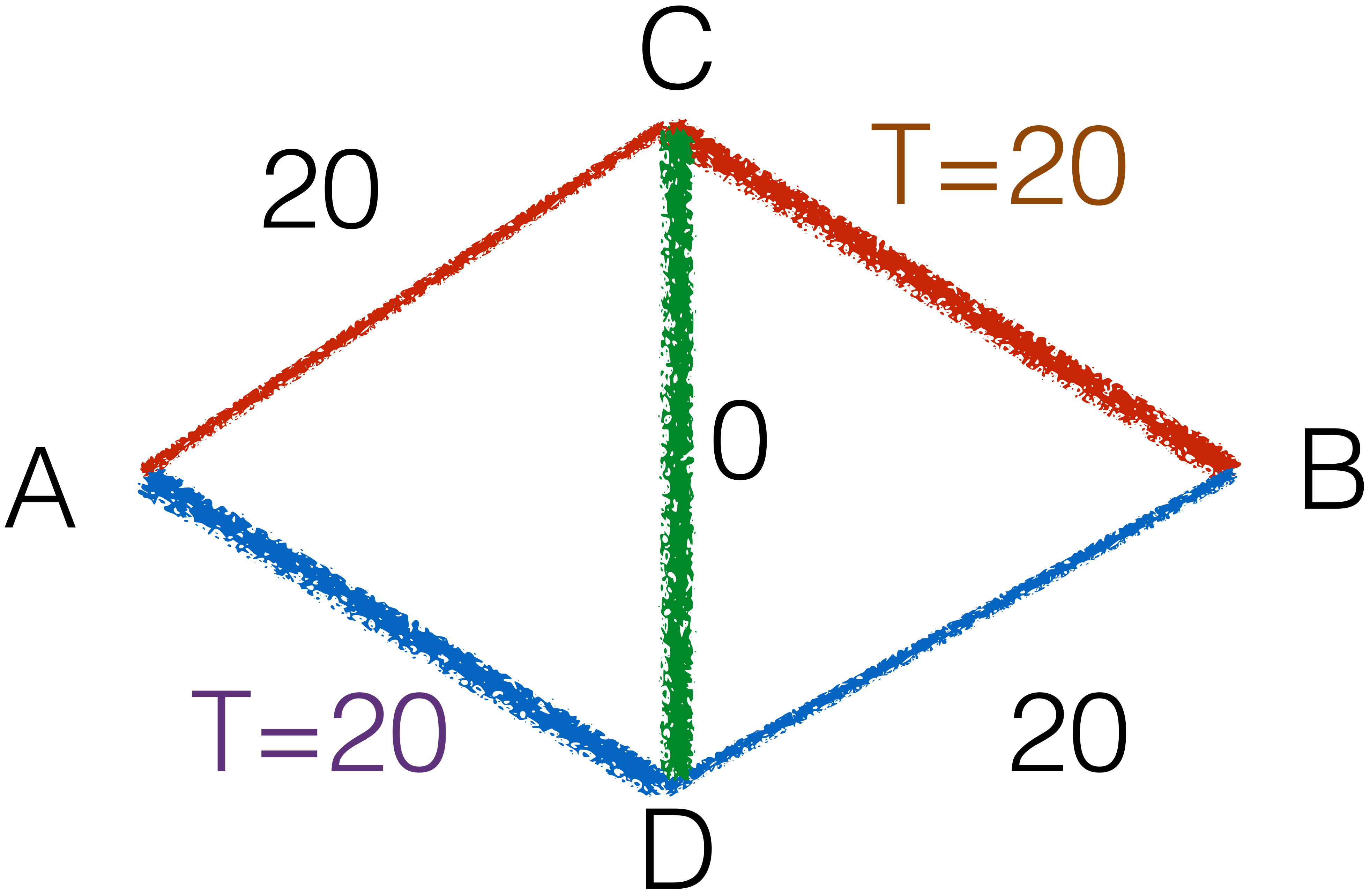
At this point, which route is more preferable?



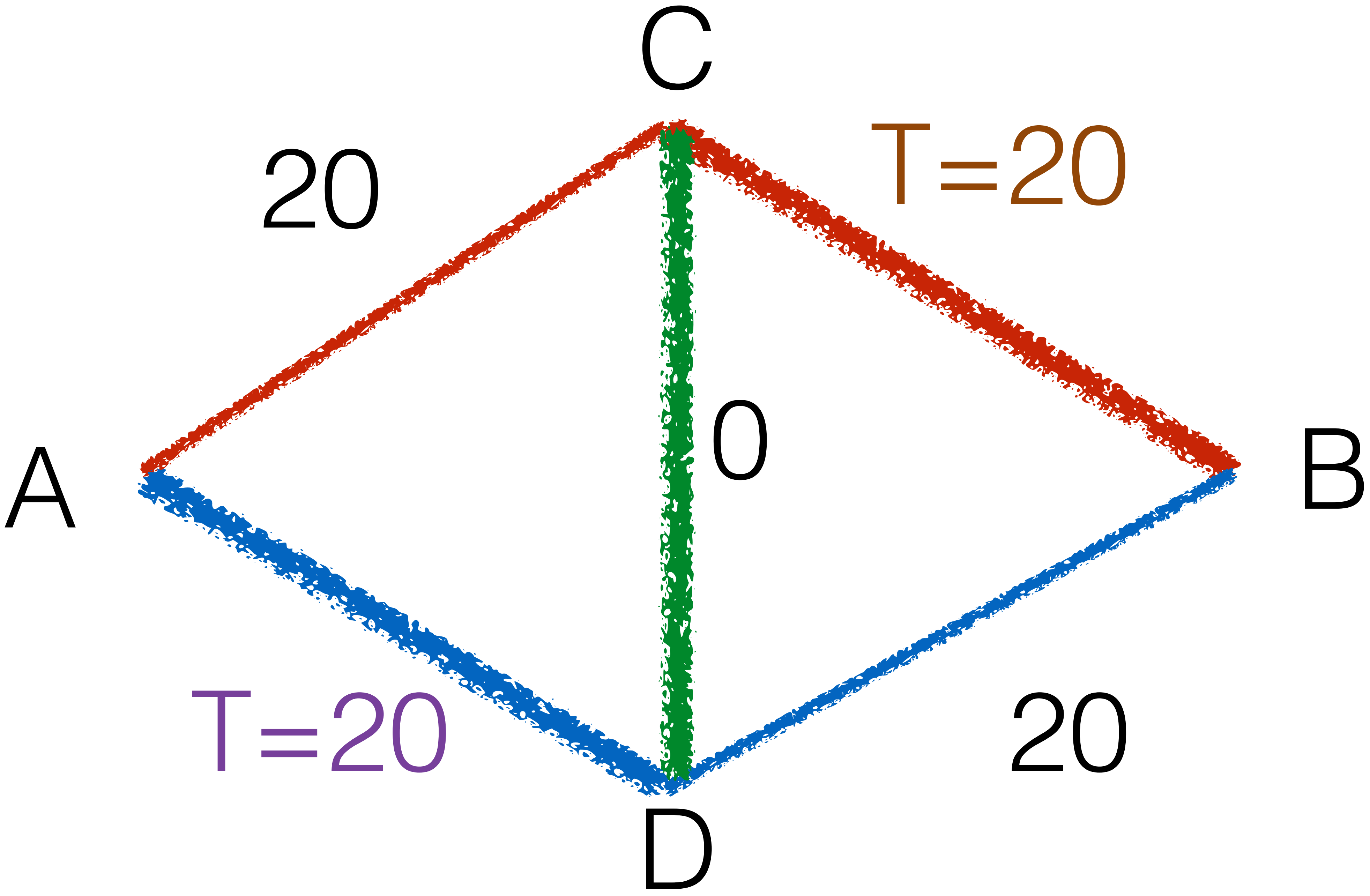
More and more cars take route A-D-C-B, until the new equilibrium is established, which is …



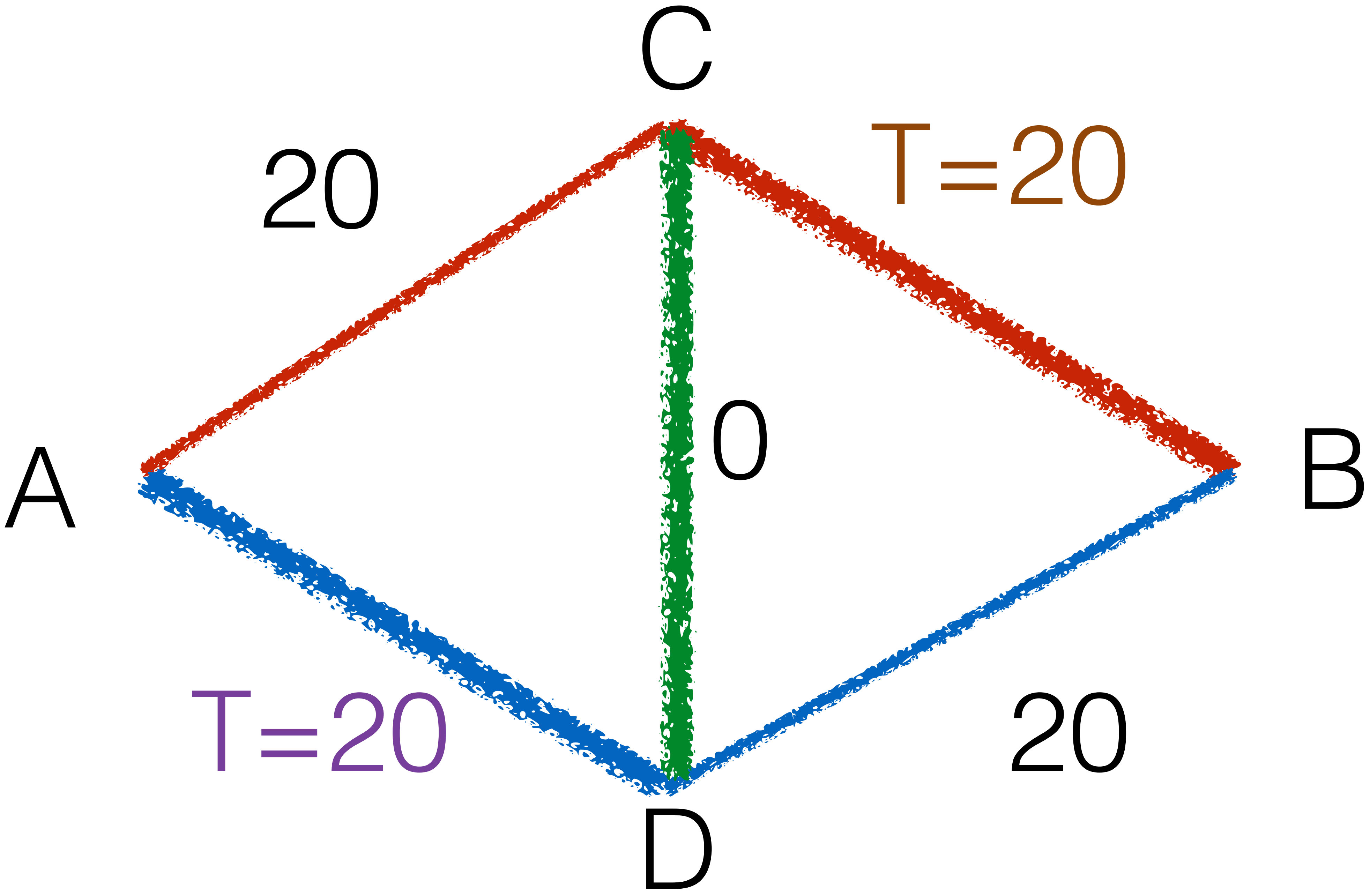
All cars choose A-D-C-B, and the travel time is 40 minutes!



Wait… we added a new road,
and the travel time increases?



The mixed strategy Nash equilibrium may not give an optimal choice !



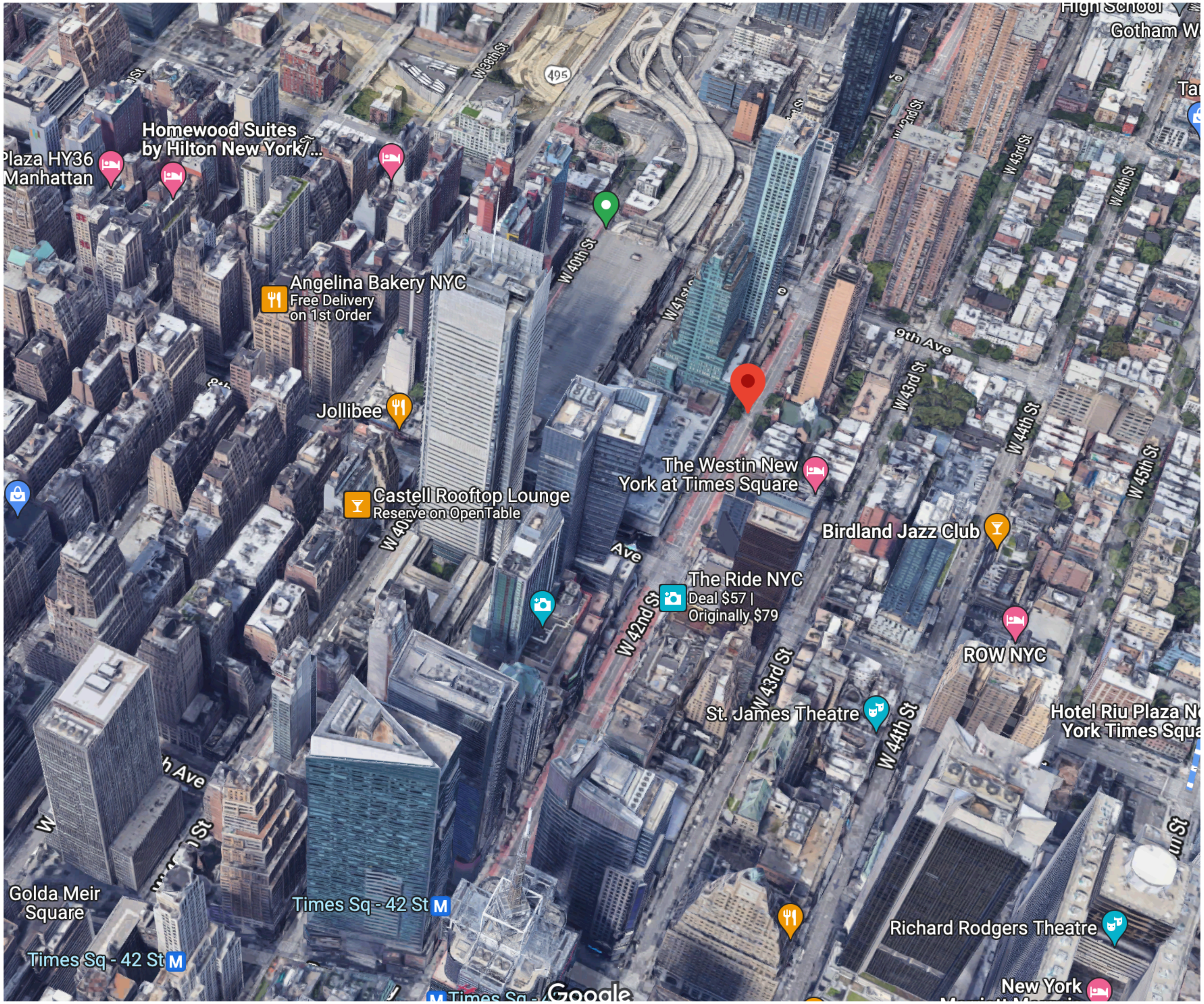
This phenomenon is called **Braess's paradox**

Braess's paradox

Braess's paradox is the observation that adding one or more roads to a road network can slow down overall traffic flow through it.

In Stuttgart, Germany, investments were made in 1969 to improve roads to decrease congestion. Congestion did not decrease until a section of newly built road was closed for traffic again.

In 1990 the temporary closing of 42nd Street in New York City for Earth Day reduced the amount of congestion in the area.



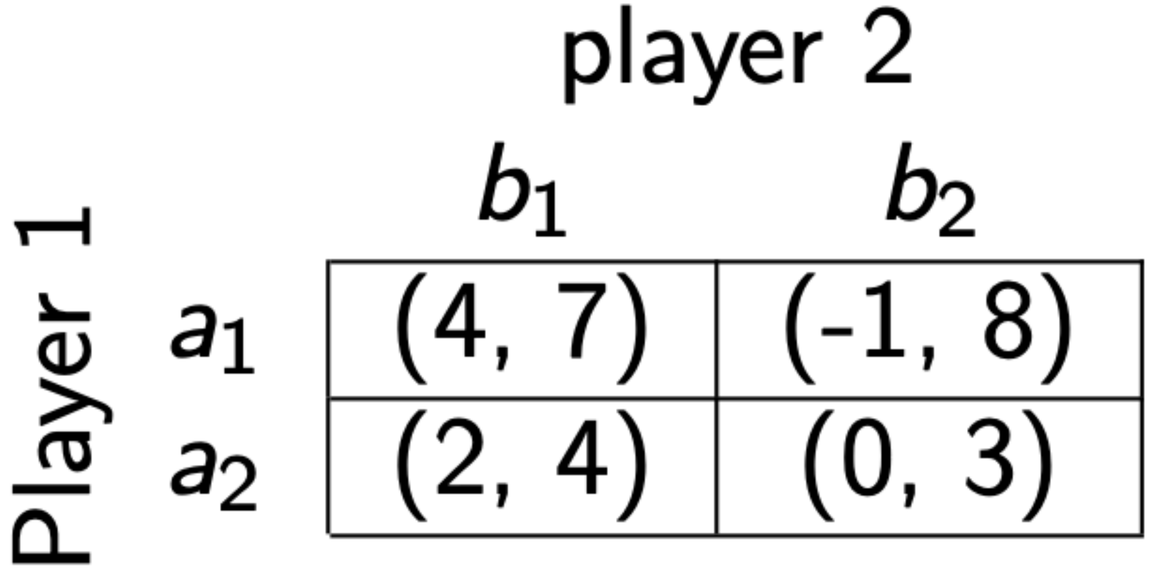
In 2009, New York experimented with closures of Broadway at Times Square and Herald Square, which resulted in improved traffic flow and permanent pedestrian plazas

How to find Mixed-strategy Nash equilibrium ?

When there are two players:

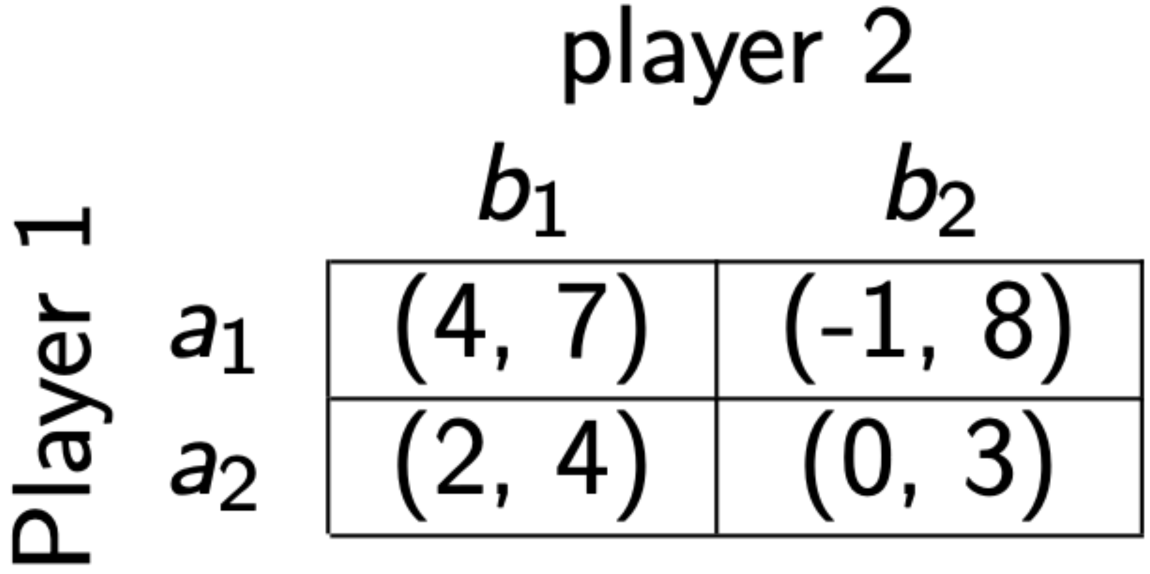
- **Step 1:** Player 1 needs to identify the mixed strategy that will make player 2’s strategies have equal payoff.
- **Step 2:** Player 2 needs to identify the mixed strategy that will make player 1’s strategies have equal payoff.

Does the following game contain any pure-strategy Nash equilibrium?



		player 2	
		b_1	b_2
Player 1	a_1	(4, 7)	(-1, 8)
	a_2	(2, 4)	(0, 3)

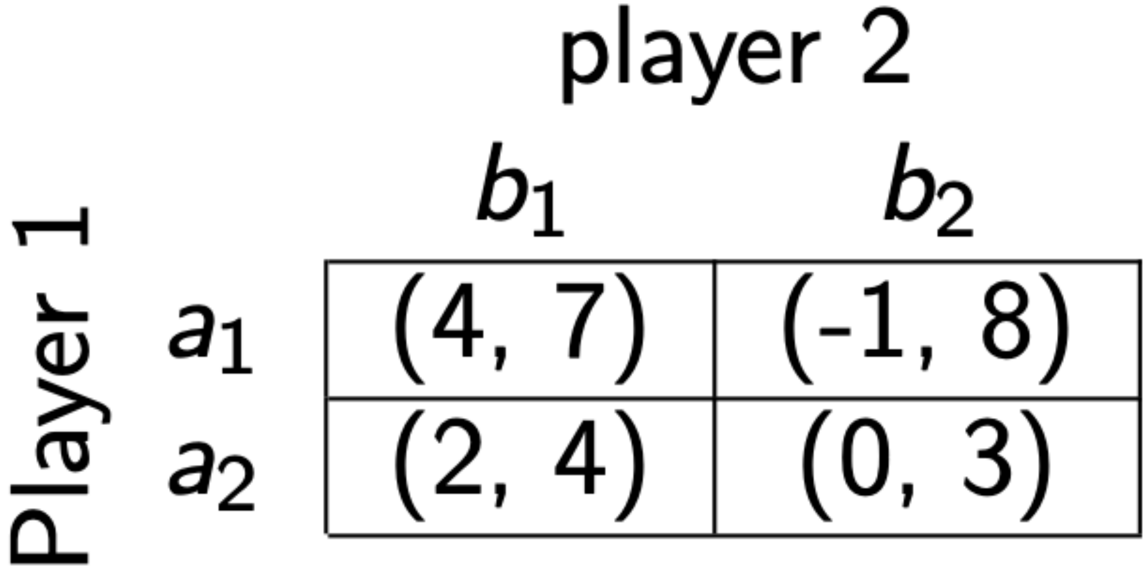
Does the following game contain any pure-strategy Nash equilibrium?



		player 2	
		b_1	b_2
Player 1	a_1	(4, 7)	(-1, 8)
	a_2	(2, 4)	(0, 3)

No!

We find the mixed-strategy Nash equilibrium



		player 2	
		b_1	b_2
Player 1	a_1	(4, 7)	(-1, 8)
	a_2	(2, 4)	(0, 3)

Step 1: We find the strategy for player 1. Assume that player 1 plays strategy a_1 with probability p and a_2 with probability $1-p$. We compute the reward of player 2 if she uses strategy b_1 and the reward of player 2 if she uses strategy b_2 .

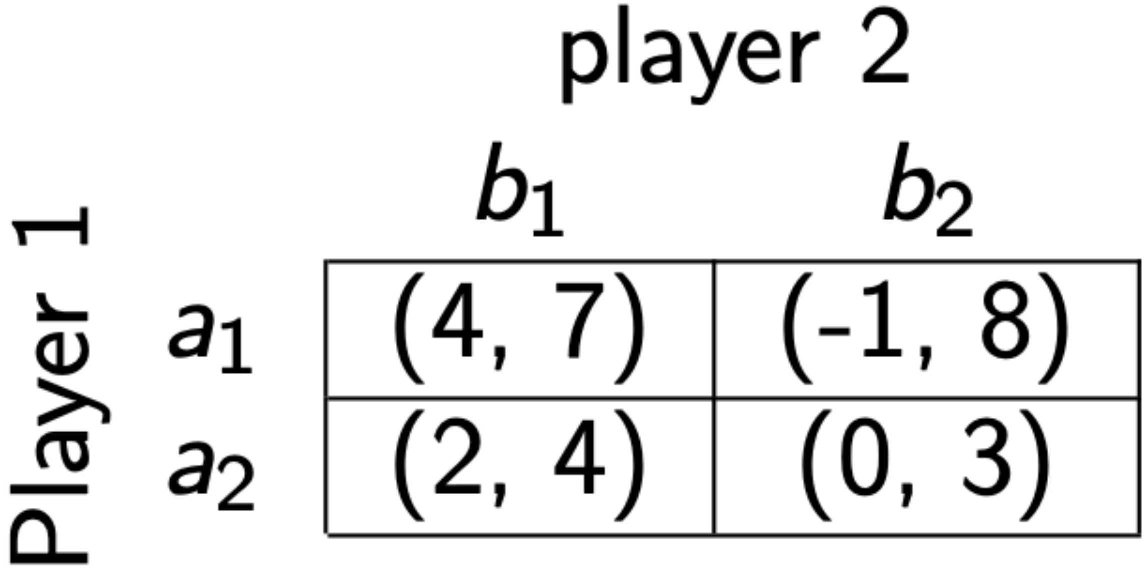
We find the mixed-strategy Nash equilibrium

		player 2	
		b_1	b_2
Player 1	a_1	(4, 7)	(-1, 8)
	a_2	(2, 4)	(0, 3)

If player 2 plays b_1, reward is $7p + 4(1-p) = 3p + 4$

If player 2 plays b_2, reward is $8p + 3(1-p) = 5p + 3$

We find the mixed-strategy Nash equilibrium



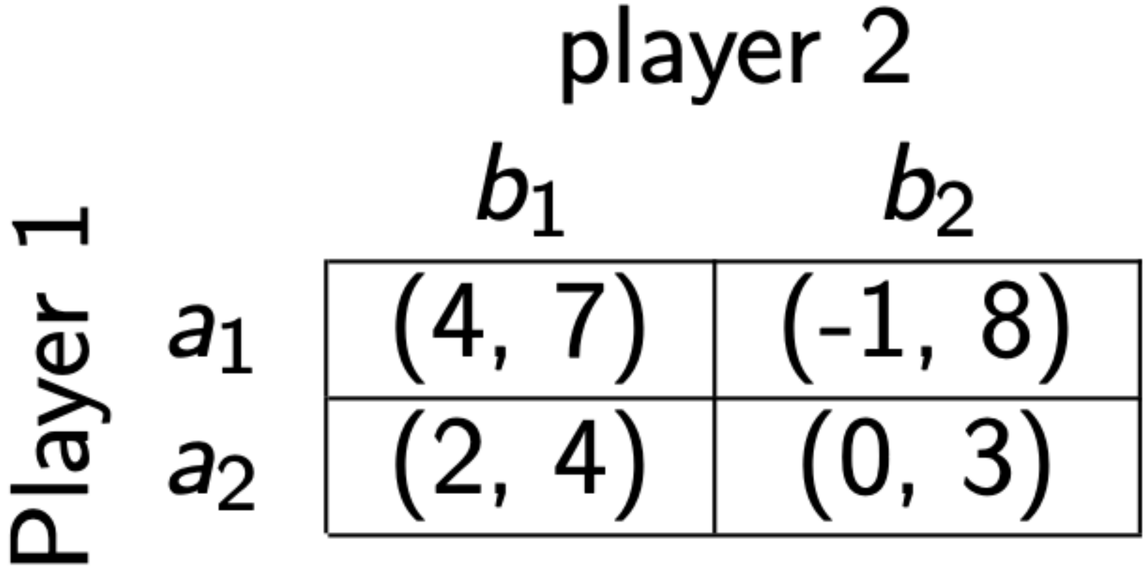
player 2

Player 1

	b_1	b_2
a_1	(4, 7)	(-1, 8)
a_2	(2, 4)	(0, 3)

We now equate the two rewards and solve for p .

We find the mixed-strategy Nash equilibrium



player 2

Player 1

	b_1	b_2
a_1	(4, 7)	(-1, 8)
a_2	(2, 4)	(0, 3)

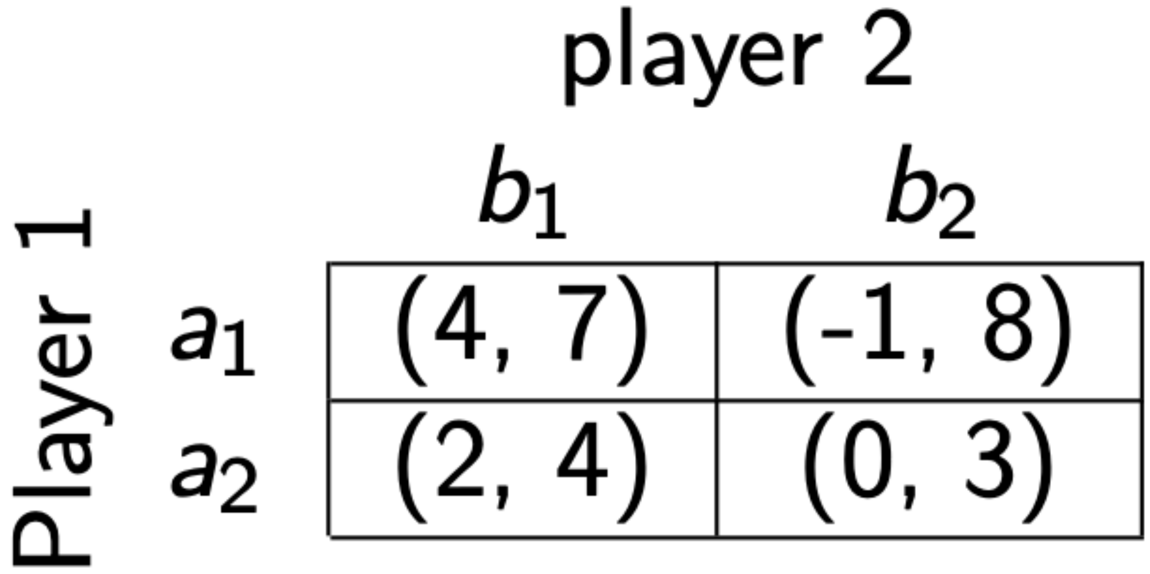
We now equate the two rewards and solve for p.

If the two rewards are equal:

$$3p + 4 = 5p + 3 \Rightarrow p = 0.5$$

This means that the strategy for player 1 in the Nash equilibrium is $pa_1 + (1-p)a_2 = 0.5a_1 + 0.5a_2.$

We find the mixed-strategy Nash equilibrium



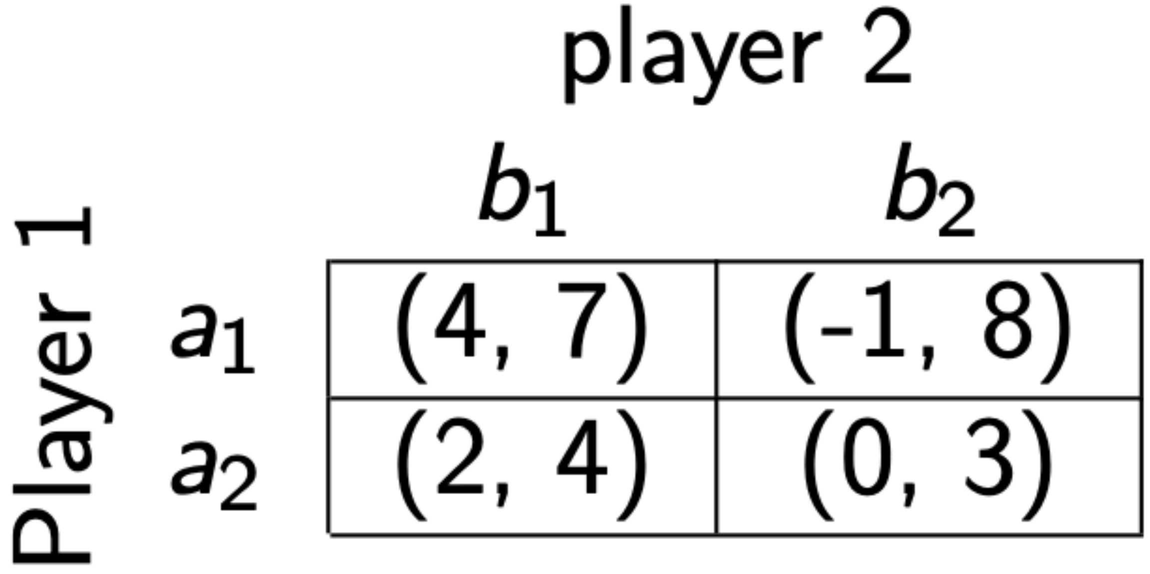
player 2

Player 1

	b_1	b_2
a_1	(4, 7)	(-1, 8)
a_2	(2, 4)	(0, 3)

Step 2: We find the strategy for player 2. Assume that player 2 plays strategy b_1 with probability q and b_2 with probability $1-q$. We compute the reward of player 1 if he uses strategy a_1 and the reward of player 1 if he uses strategy a_2 .

We find the mixed-strategy Nash equilibrium



player 2

Player 1

	b_1	b_2
a_1	(4, 7)	(-1, 8)
a_2	(2, 4)	(0, 3)

If player 1 plays a_1, reward is $4q - (1-q) = 5q - 1$

If player 1 plays a_2, reward is $2q + 0(1-q) = 2q$

We find the mixed-strategy Nash equilibrium

		player 2	
		b_1	b_2
Player 1	a_1	(4, 7)	(-1, 8)
	a_2	(2, 4)	(0, 3)

We now equate the two rewards and solve for q .

If the two rewards are equal:

$$5q - 1 = 2q \Rightarrow q = 1/3$$

This means that the strategy for player 2 in the Nash equilibrium is

$$qb_1 + (1-q)b_2 = \frac{1}{3}b_1 + \frac{2}{3}b_2.$$

player 2

Player 1

	b_1	b_2	$\frac{b_1}{3} + \frac{2b_2}{3}$
a_1	(4, 7)	(-1, 8)	(2/3, 23/3)
a_2	(2, 4)	(0, 3)	(2/3, 10/3)
$\frac{a_1}{2} + \frac{a_2}{2}$	(3, 5.5)	(-0.5, 5.5)	(2/3, 5.5)

Nash equilibrium

player 2

		b_1	b_2	$\frac{b_1}{3} + \frac{2b_2}{3}$
Player 1	a_1	(4, 7)	(-1, 8)	(2/3, 23/3)
	a_2	(2, 4)	(0, 3)	(2/3, 10/3)
	$\frac{a_1}{2} + \frac{a_2}{2}$	(3, 5.5)	(-0.5, 5.5)	(2/3, 5.5)

Nash equilibrium