Probability - Part 1

Jan. 28, 2025

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By the end of this lecture, you will be able to:

1. Define probability and sample space, disjoint events and independent events
2. Compute probabilities of rolling dice and drawing cards

“The probability of rain is 70% this afternoon.”

“Primary efficacy analysis demonstrates that X vaccine to be 95% effective against Y disease beginning 28 days after the first dose.”

“The chance that the US stock market crashes this year is 99%.”

Commonality?

For today: flipping coins, rolling dice, drawing cards

Tom Dwan

Tom Dwan on *High Stakes Poker* in 2008

Wiki

Texas Hold’em

Random events: the individual outcomes are uncertain, but the long-term pattern of many individual outcomes is predictable. Example: heads or tails when tossing a fair coin.

Probability: the proportion of times the outcome occurs over a long series of repetitions.

$$P(A) = \frac{\#\text{ of outcomes in } A}{\#\text{ of possible outcomes}}$$

Sample space: set of all possible outcomes

Example 1: If you toss a fair coin, the sample space is {H, T}.

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Example 2: What is the sample space when two fair dice are tossed?

{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), …, (6,1), (6,2), (6,3), (6,4), (6, 5), (6, 6)}

First roll   Second roll

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What is the sample space for the sum of the two rolls?

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What is the sample space for the sum of the two rolls?

{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}.

Sample space:

{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),…, (6,1), (6,2), (6,3), (6, 4), (6, 5), (6, 6)}

Sample space:

{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),…, (6,1), (6,2), (6,3), (6, 4), (6, 5), (6, 6)}

P(sum = 2) = 1 / 36

P(sum = 3) = 2 / 36

P(sum = 4) = 3 / 36

…

P(sum = 12) = 1 / 36

Sample space:

{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),…, (6,1), (6,2), (6,3), (6, 4), (6, 5), (6, 6)}

P(sum = 2) = 1 / 36

P(sum = 3) = 2 / 36

P(sum = 4) = 3 / 36

…

P(sum = 12) = 1 / 36

P(sum < 6) =

P(sum is odd) =

P(same number on both dice) =

Sample space:

{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),…, (6,1), (6,2), (6,3), (6, 4), (6, 5), (6, 6)}

P(sum = 2) = 1 / 36

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P(sum = 4) = 3 / 36

…

P(sum = 12) = 1 / 36

P(sum < 6) = 10/36

P(sum is odd) = 18/36

P(same number on both dice) = 6/36

Two properties

1. Every probability is a number between 0 and 1.

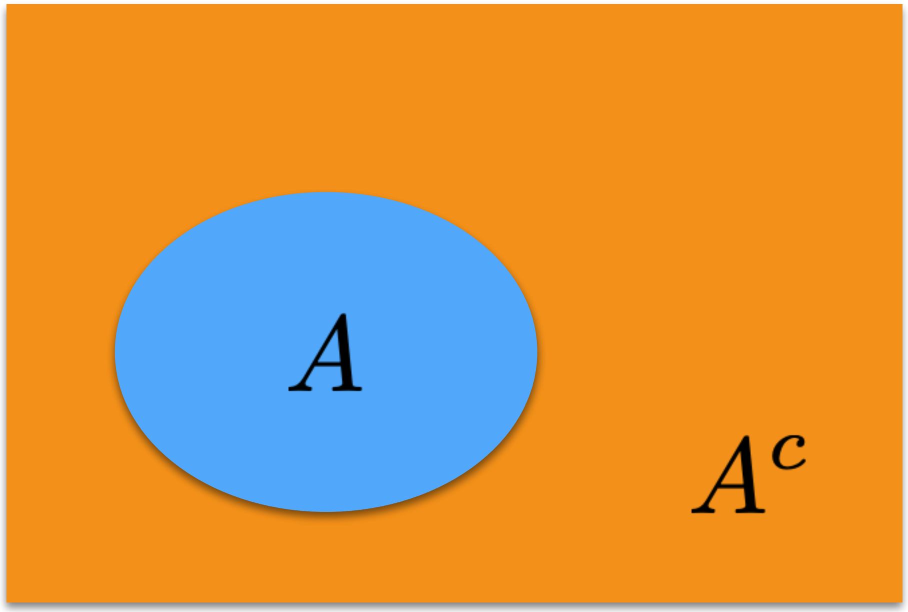
2. The sum of probabilities over all possible outcomes is 1.

Example:

P(sum = 2) +P(sum=3) + …+P(sum=12) = 1.

More definitions

Complement: The complement of event A , denoted by A^c, is the set of all outcomes that are not in the event A



$$P(A) + P(A^c) = 1$$

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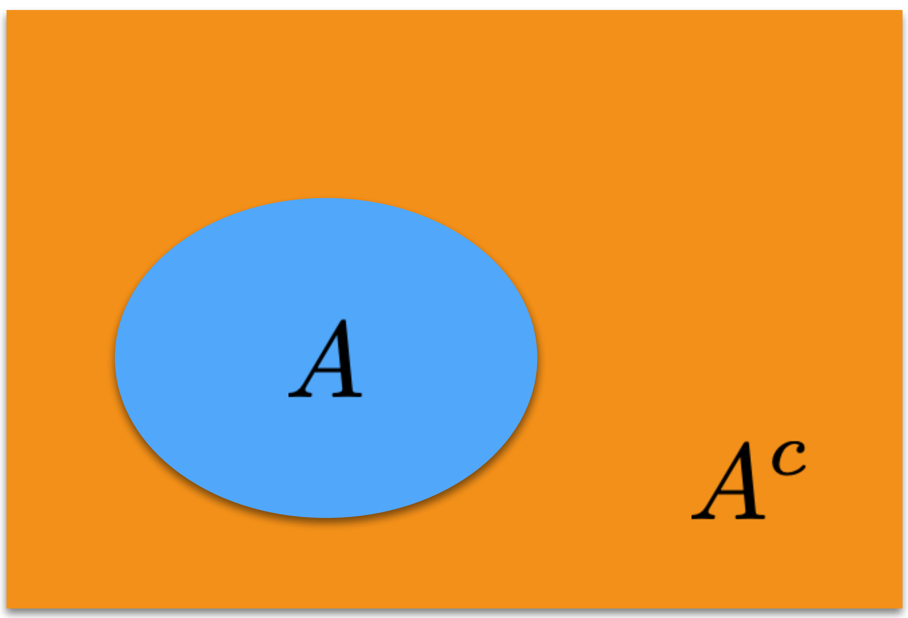
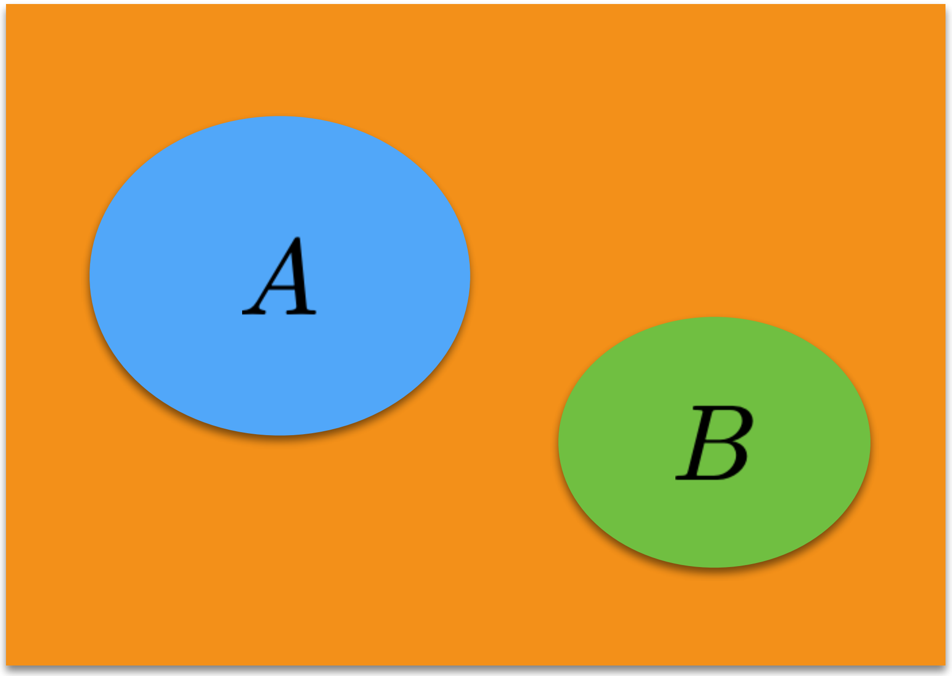


$$P(A) + P(A^c) = 1$$

Example: If A = roll 1, then A^c = roll 2,3,4,5, **or** 6

Disjoint (mutually exclusive): events that cannot happen at the same time.



$$P(A \text{ and } B) = 0$$
$$P(A \text{ or } B) = P(A) + P(B)$$

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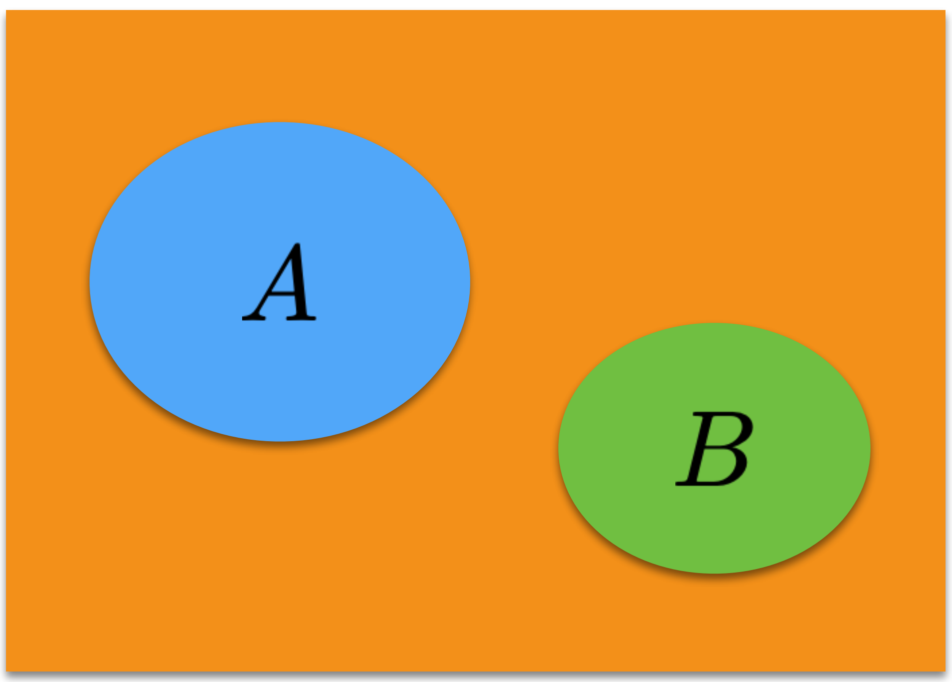

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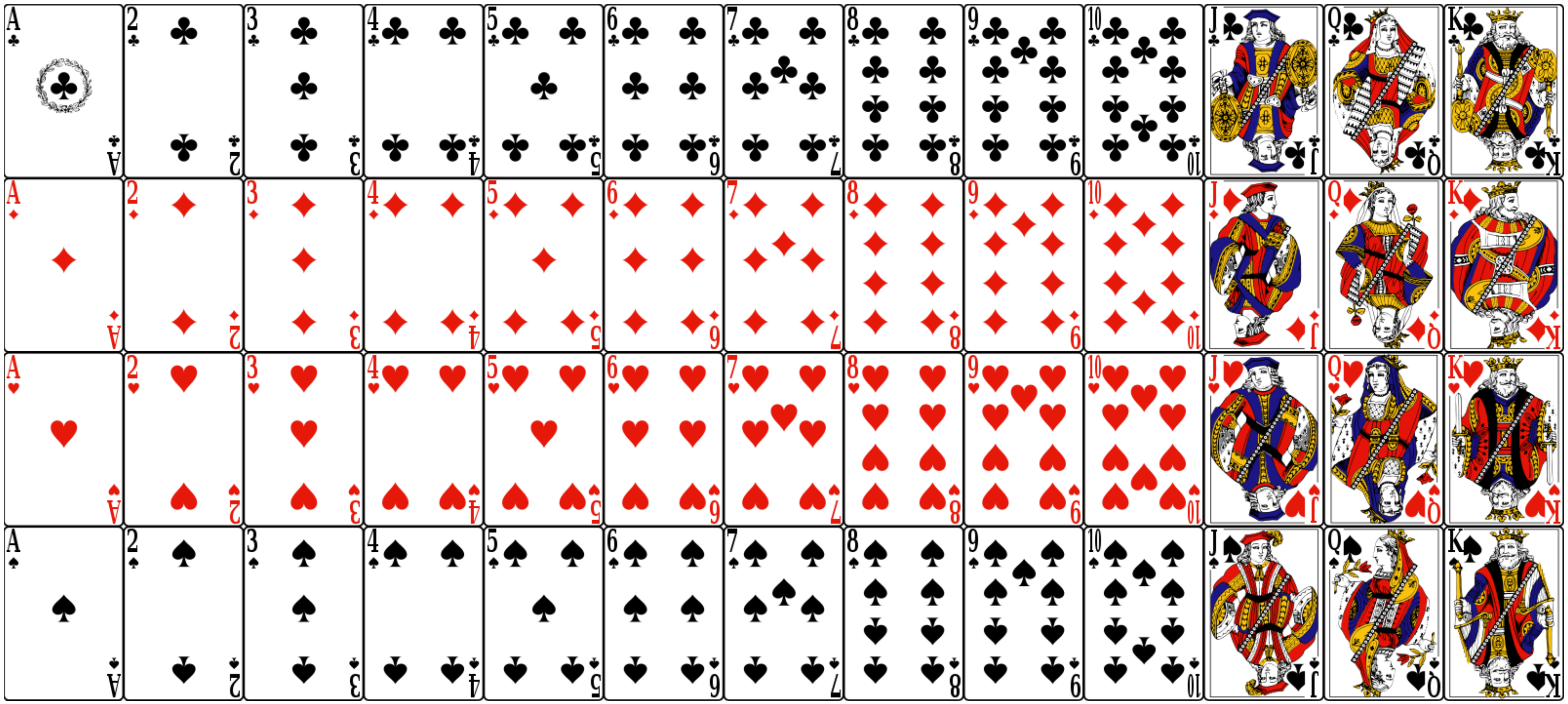
$$P(A \text{ or } B) = P(A) + P(B)$$

Example: A = roll 1, B = roll 2



13 cards

13 cards

13 cards

13 cards

Total: 52 cards

Example 1: Drawing a card from a deck of cards,

A = queen, B = king

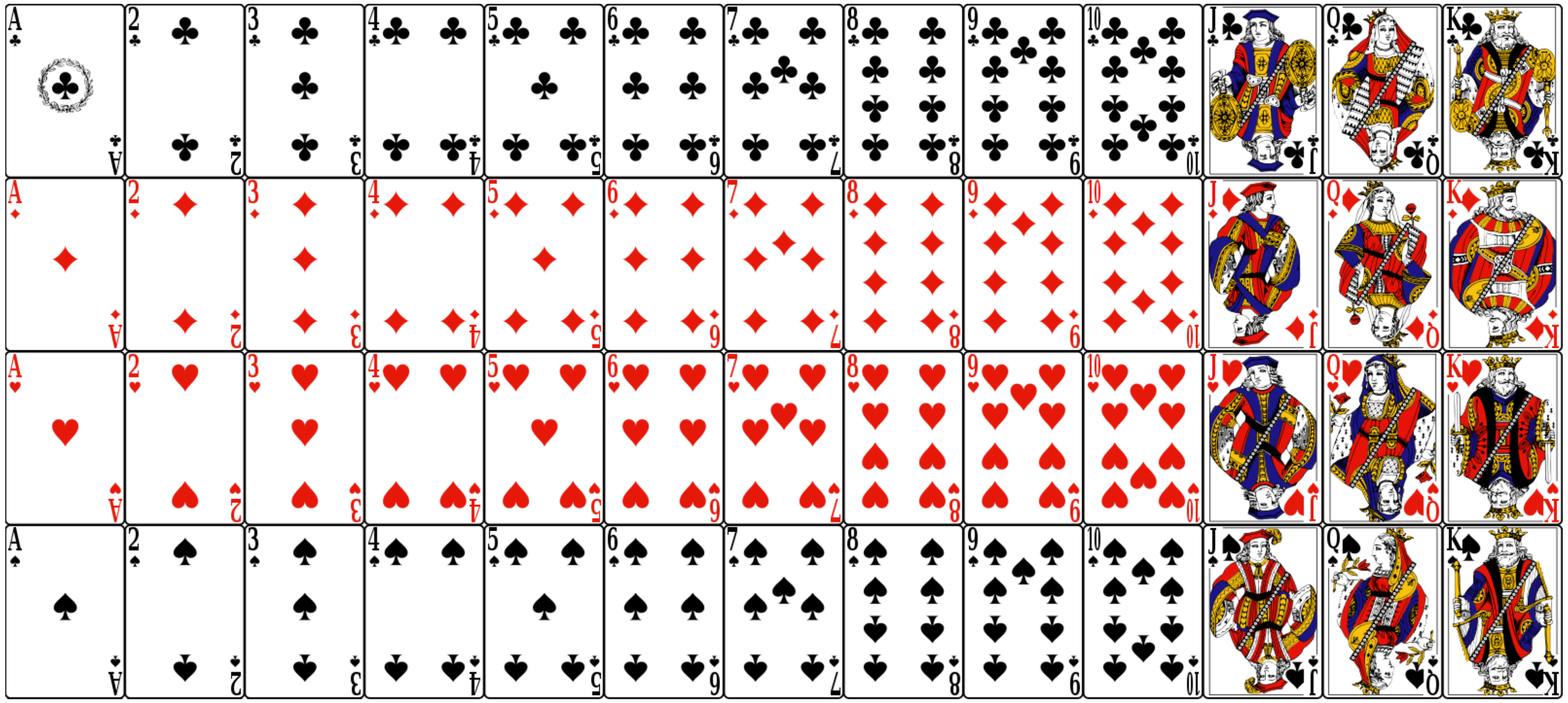
Disjoint events!

P(A) =

P(B) =

P(A and B) =

P(A or B) =



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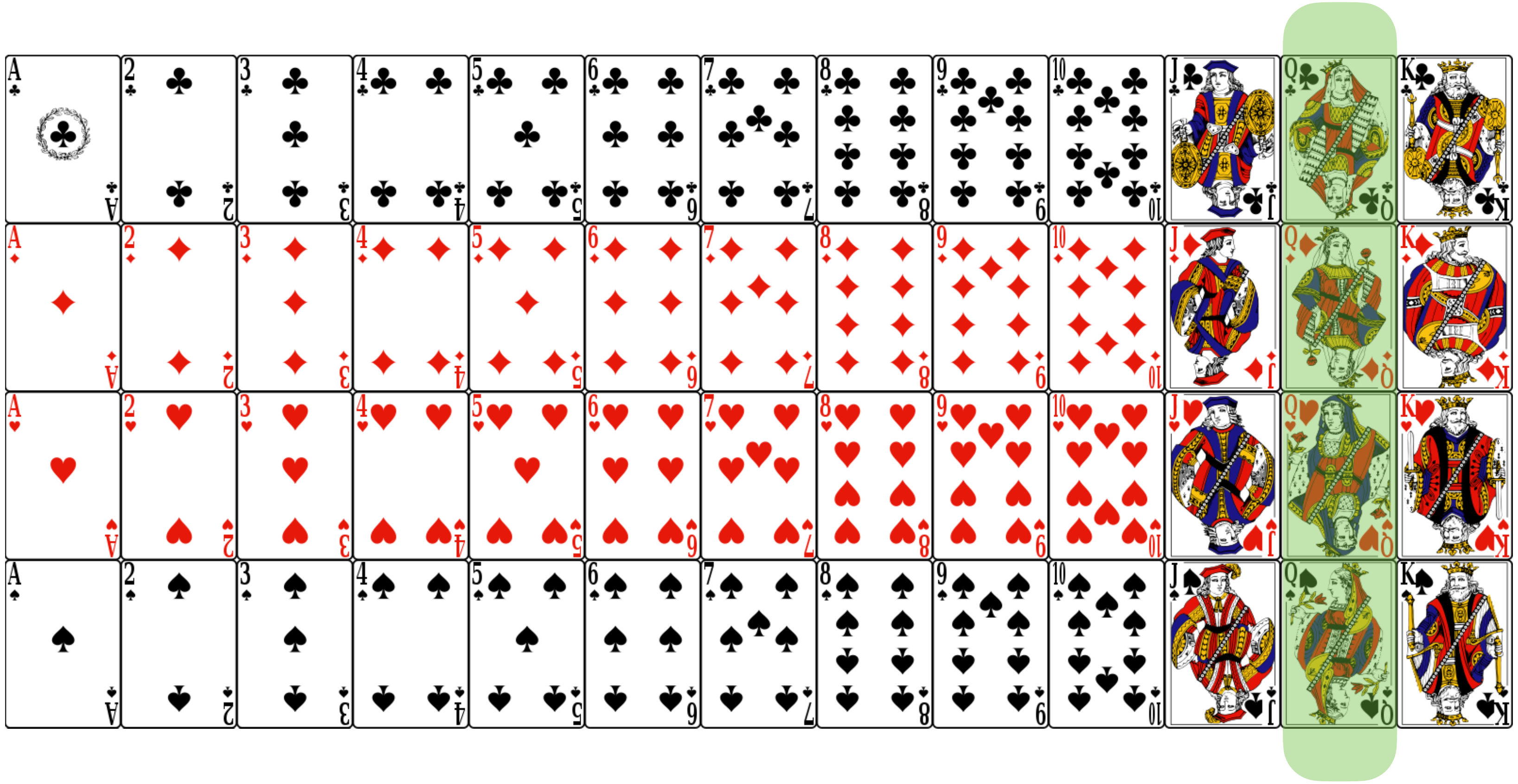


Example 1: Drawing a card from a deck of cards,

A = queen, B = king

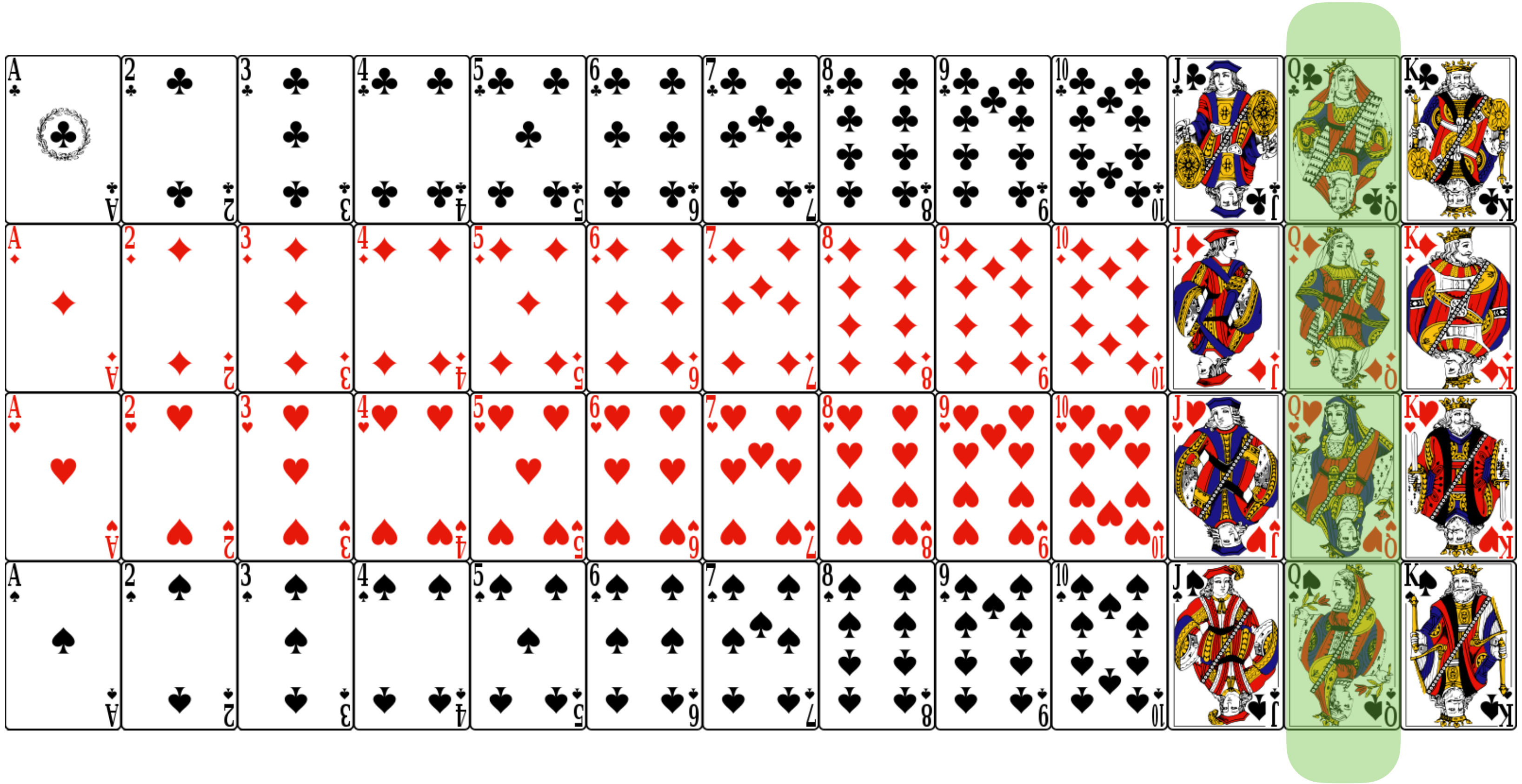
Disjoint events!

P(A) = 4/52

P(B) =

P(A and B) =

P(A or B) =



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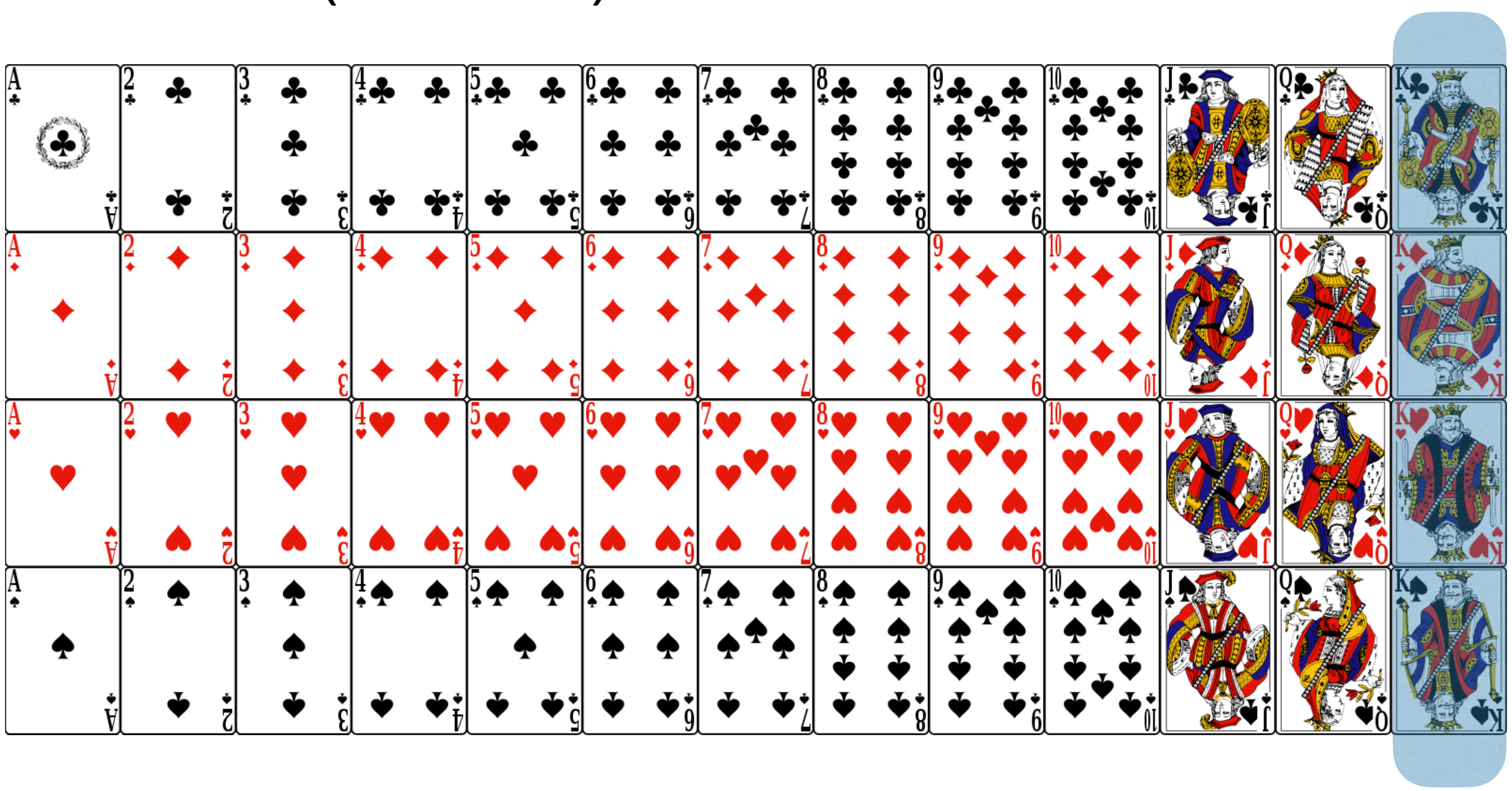

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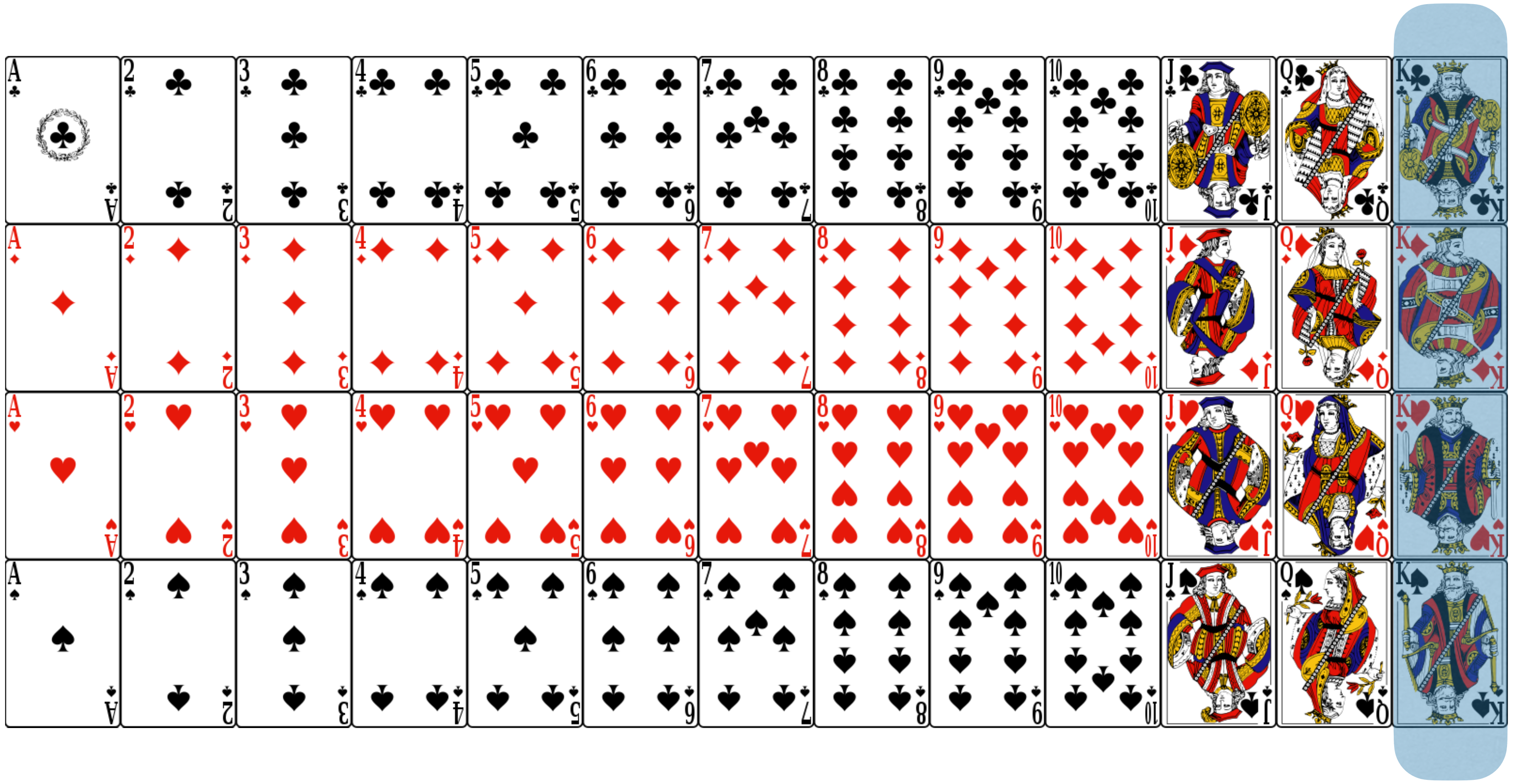

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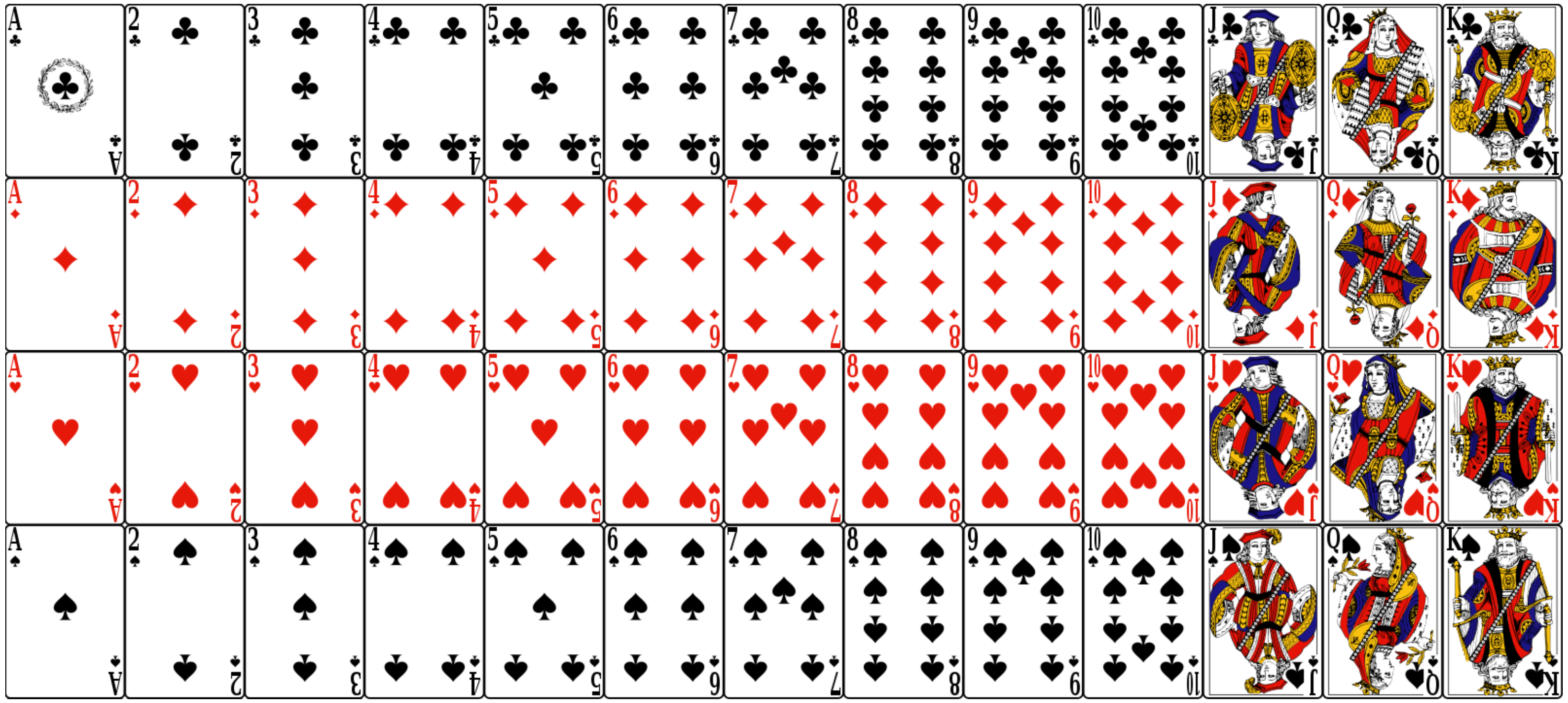

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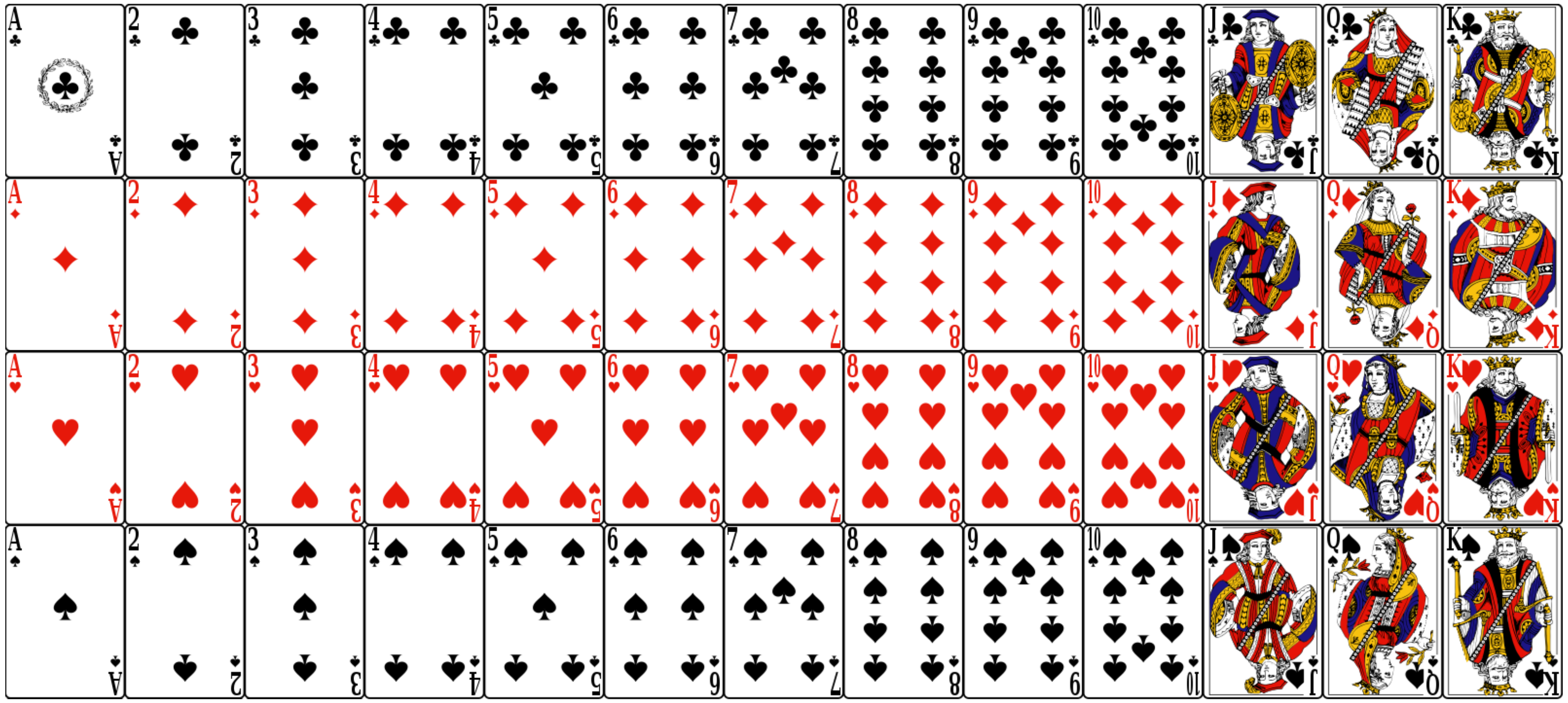


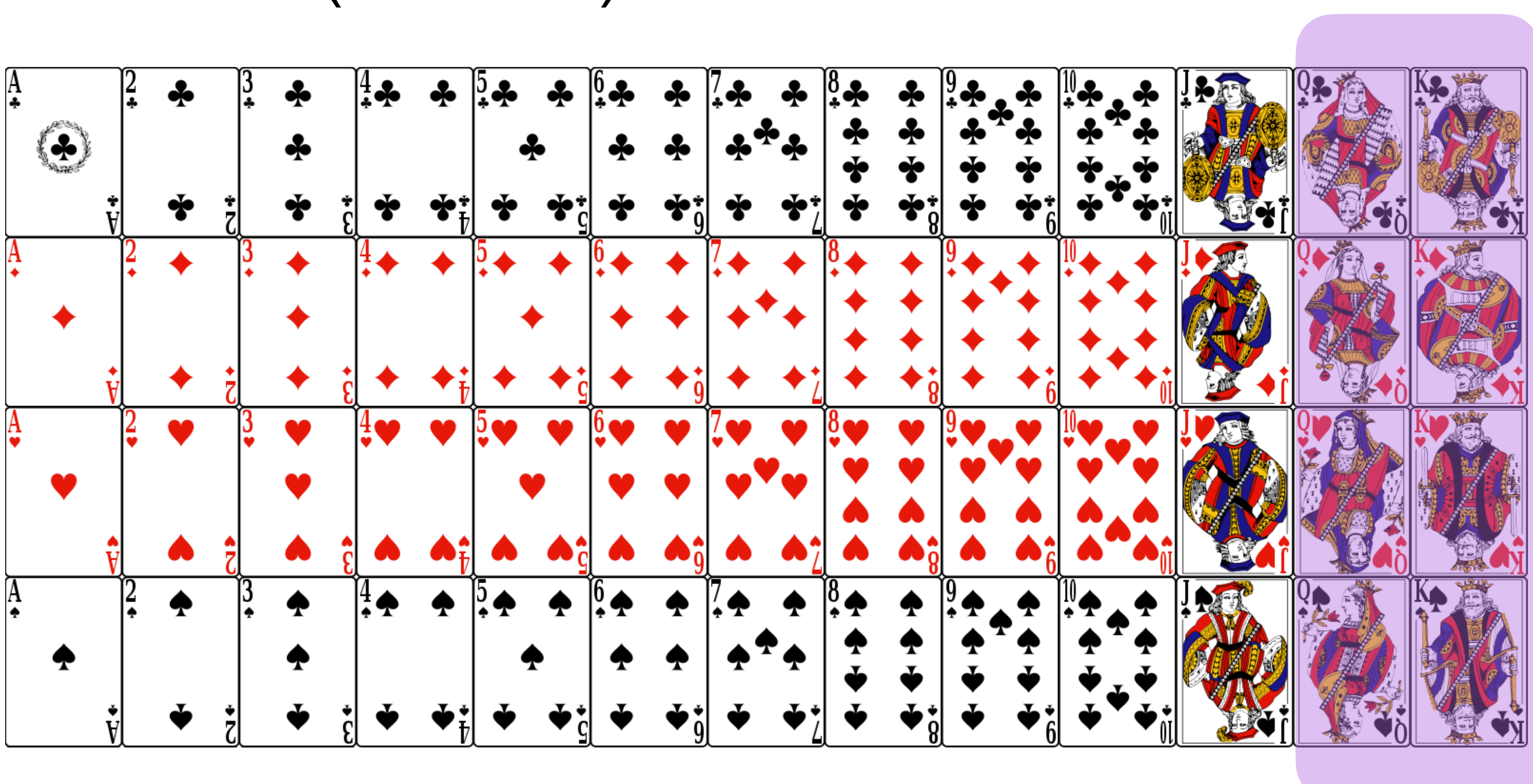
Example 1: Drawing a card from a deck of cards,

A = queen, B = king

Disjoint events!

P(A) = 4/52
P(B) = 4/52
P(A and B) = 0

P(A or B) = 8/52



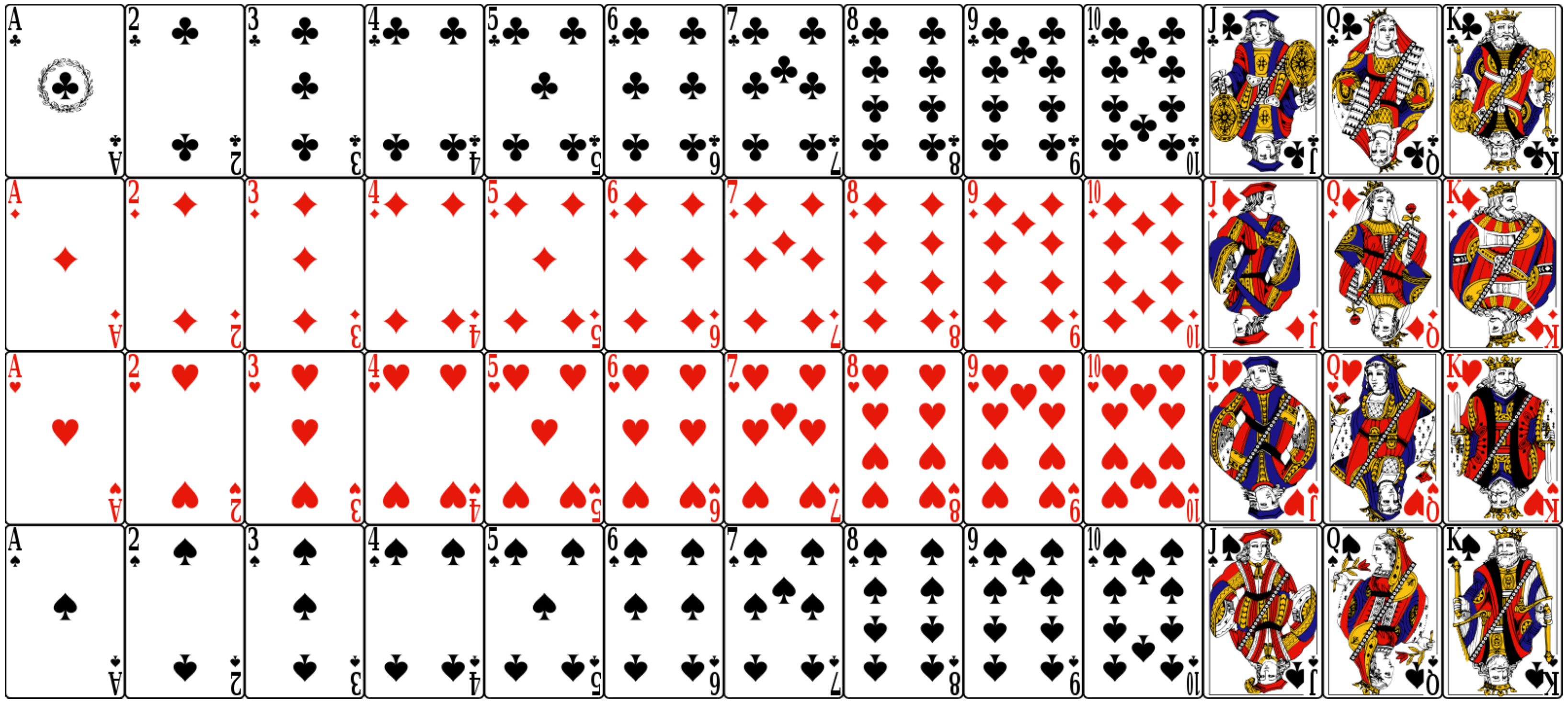
Example 2: Drawing a card from a deck of cards,

A = queen, B = diamond suit

Not disjoint events

P(A and B) =

P(A or B) =



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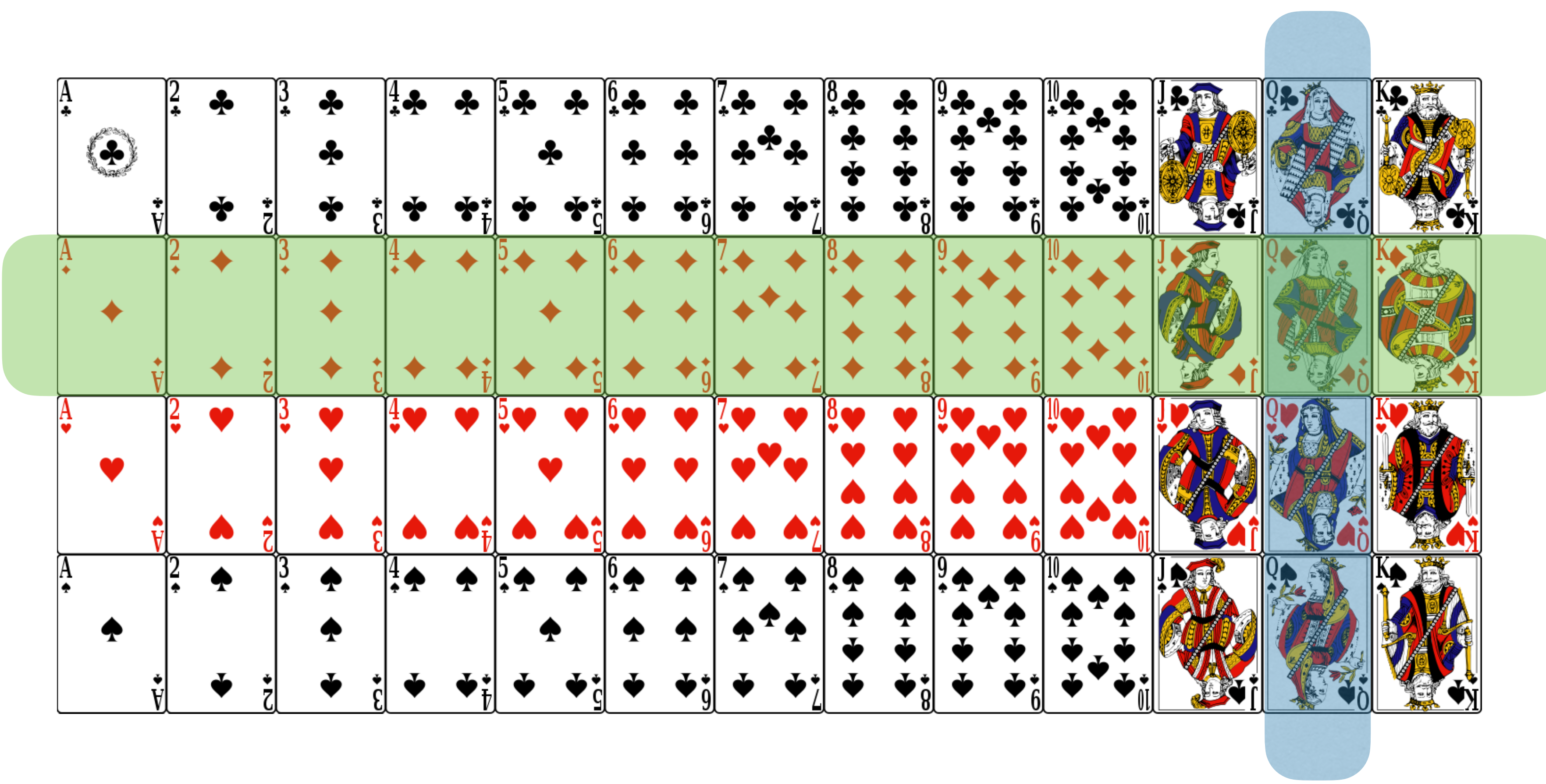
Not disjoint events

P(A and B) =

P(A or B) =



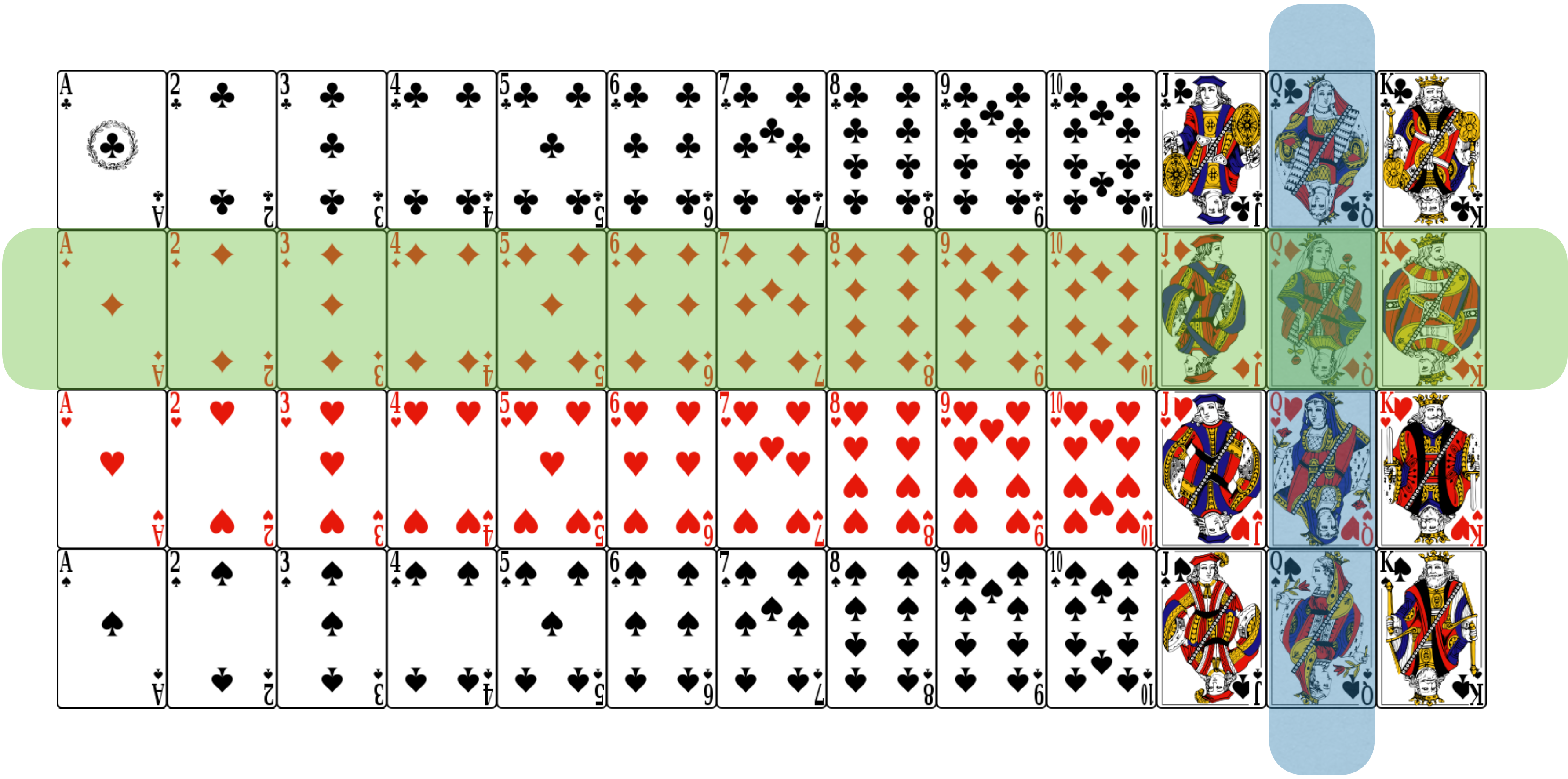
Example 2: Drawing a card from a deck of cards,

A = queen, B = diamond suit

Not disjoint events

P(A and B) = 1/52

P(A or B) = 16/52



Independent: two events are independent if the outcome of the first event does not affect the outcome of the second.

Example 1: If you toss a fair coin for 10 times and get 10 heads, the probability to get a head at the 11-th time is still 1/2.

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Example 2: Drawing two cards from a deck of cards (and remove the drawn from the deck), A = get a queen in the first draw, B = get a **king** in the second draw

Conditional Probability

P(A|B) = the probability that A occurs, given that B has happened

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

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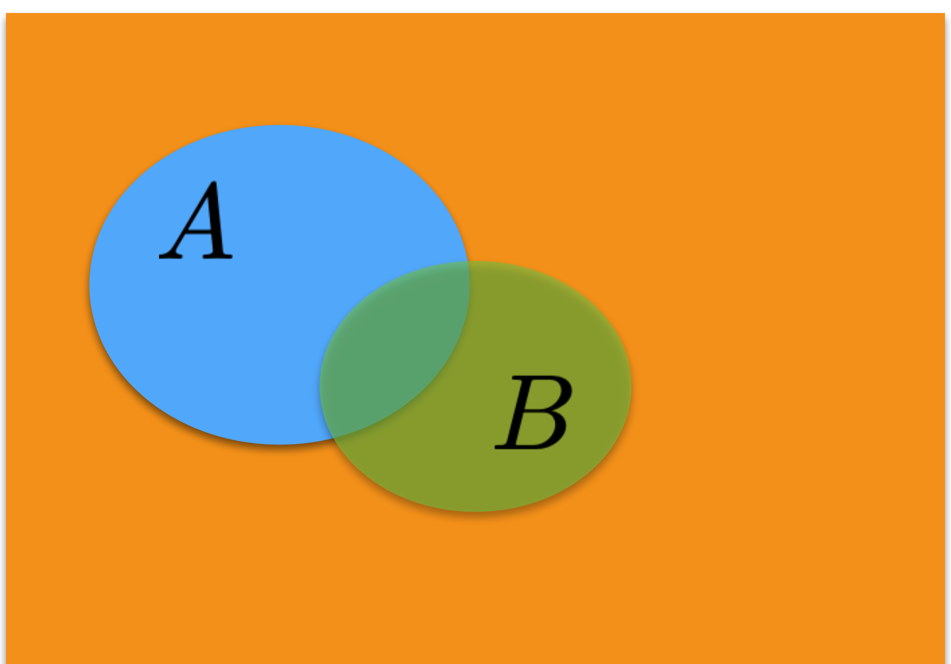

If A and B are independent, P(A|B) = P(A).

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Example 1:

When rolling two dice,

A: obtain 2 on one of the rolls

B: the sum equals 7

P(A) =

P(B) =

P(A|B) =

P(B|A) =

P(A and B) =

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

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Example 1:

When rolling two dice,

A: obtain 2 on one of the rolls

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P(A) = 11/36

P(B) =

P(A|B) =

P(B|A) =

P(A and B) =

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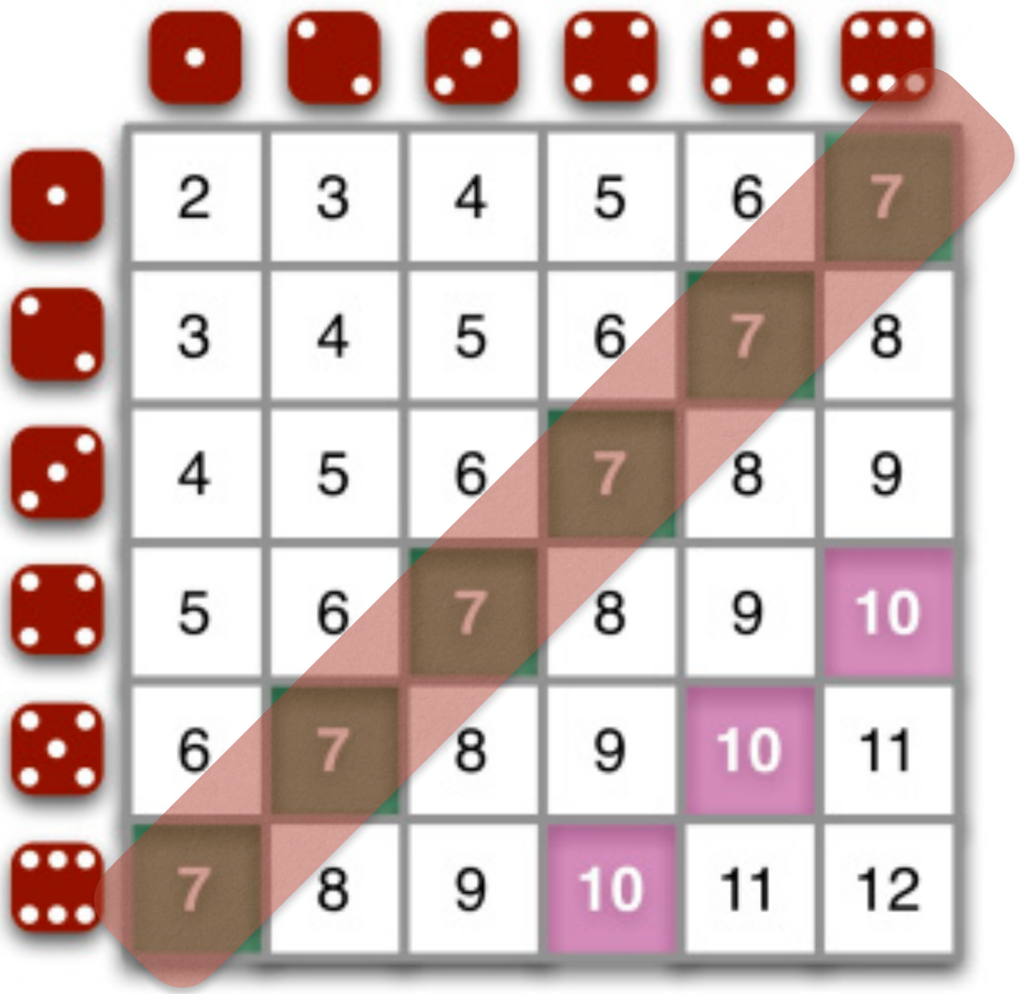

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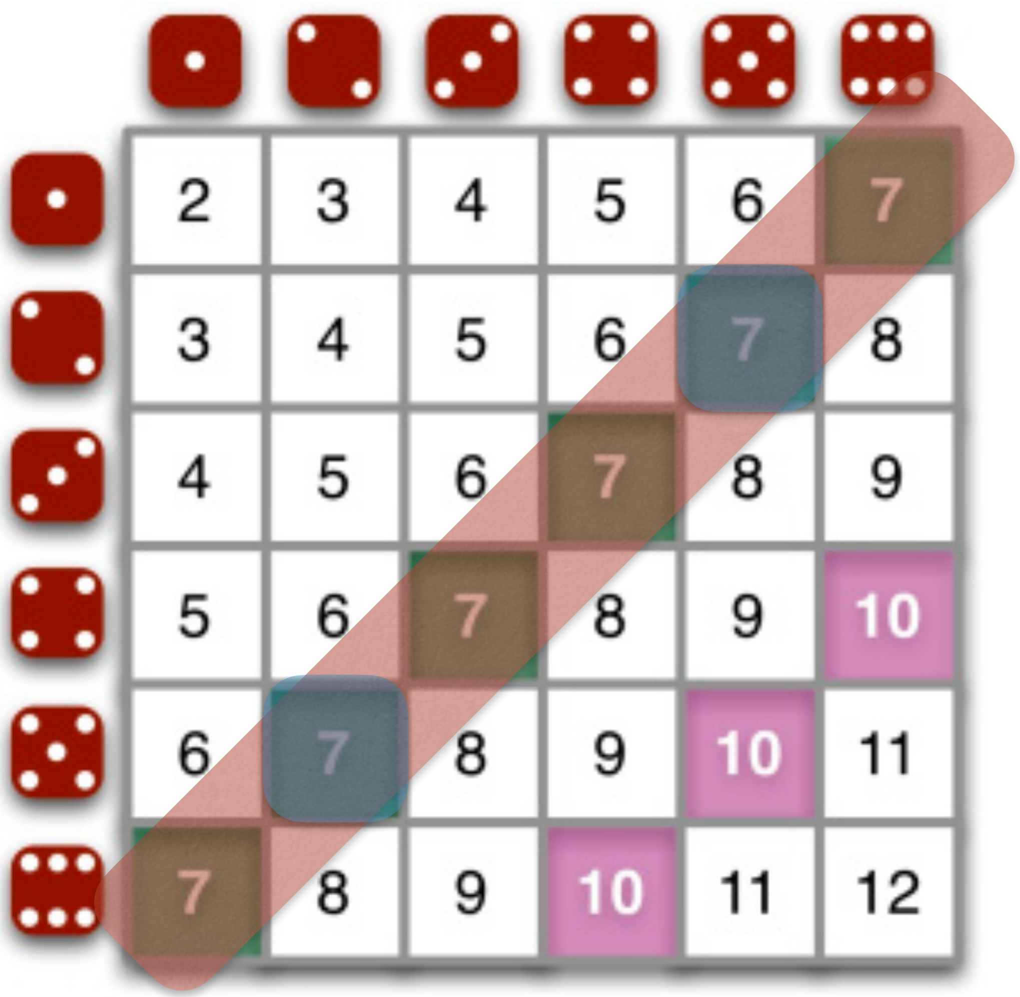

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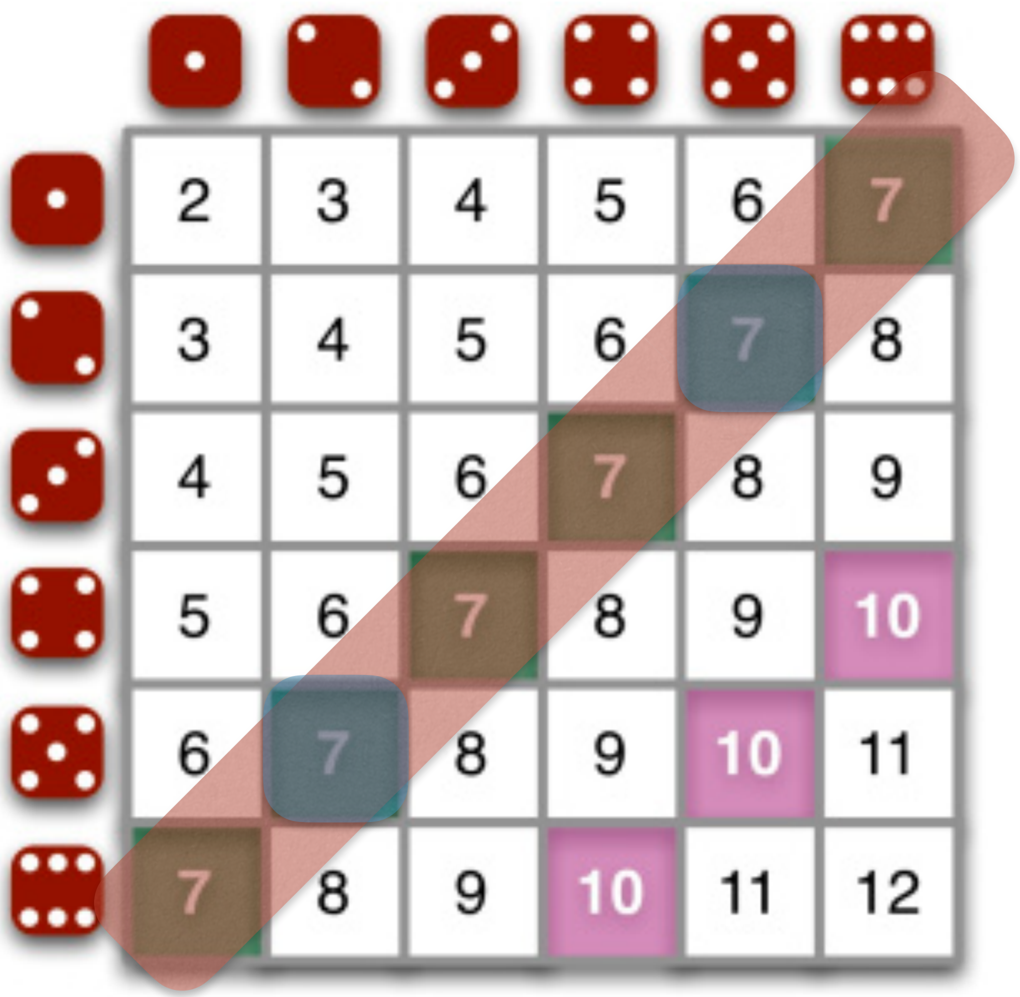

Example 1:

When rolling two dice,

A: obtain 2 on one of the rolls

B: the sum equals 7

P(A) = 11/36

P(B) = 6/36

P(A|B) = 2/6

P(B|A) =

P(A and B) =

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

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P(A and B) =

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P(A) = 11/36

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P(A and B) =

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

	⚀	⚁	⚂	⚃	⚄	⚅
⚀	2	3	4	5	6	7
⚁	3	4	5	6	7	8
⚂	4	5	6	7	8	9
⚃	5	6	7	8	9	10
⚄	6	7	8	9	10	11
⚅	7	8	9	10	11	12

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When rolling two dice,

A: obtain 2 on one of the rolls

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P(A) = 11/36

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P(A|B) = 2/6

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P(A and B) = 2/36

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

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When rolling two dice,

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P(A) = 11/36

P(B) = 6/36

P(A|B) = 2/6

P(B|A) = 2/11

P(A and B) = 2/36 = P(A|B)*P(B)

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

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When rolling two dice,

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P(A and B) = 2/36 = P(A|B)*P(B)

= P(B|A)*P(A)

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Example 2: Drawing two cards from a deck of cards, the probability that the first is a queen and the second is a king

A = draw a king in second draw
B = draw a queen in first draw
Calculate P(A and B)!

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Example 2: Drawing two cards from a deck of cards, the probability that the first is a queen and the second is a king

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Calculate P(A and B)!

$$P(A \text{ and } B) = P(A|B)P(B)$$

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

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A = draw a king in second draw
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Calculate P(A and B)!

$$P(A \text{ and } B) = P(A|B)P(B)$$

4 queens in 52 cards

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Example 2: Drawing two cards from a deck of cards, the probability that the first is a queen and the second is a king

A = draw a king in second draw
B = draw a queen in first draw
Calculate P(A and B)!

$$P(A \text{ and } B) = P(A|B)P(B)$$

4 queens in 52 cards ($P(B)$)

4 kings in 51 cards ($P(A|B)$)

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Example 2: Drawing two cards from a deck of cards, the probability that the first is a queen and the second is a king

A = draw a king in second draw
B = draw a queen in first draw
Calculate P(A and B)!

$$P(A \text{ and } B) = P(A|B)P(B) = \frac{4}{51} \cdot \frac{4}{52}$$

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Example 3: Drawing two cards from a deck of cards, the probability of drawing a diamond after first drawing a queen

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Example 3: Drawing two cards from a deck of cards, the probability of drawing a diamond after first drawing a queen

A = draw a diamond in second draw
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Calculate P(A|B)!

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Example 3: Drawing two cards from a deck of cards, the probability of drawing a diamond after first drawing a queen

A = draw a diamond in second draw
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Calculate P(A|B)!

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Example 3: Drawing two cards from a deck of cards, the probability of drawing a diamond after first drawing a queen

A = draw a diamond in second draw
B = draw a queen in first draw
Calculate P(A|B)!

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

$$P(B) = \frac{4}{52}$$

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Example 3: Drawing two cards from a deck of cards, the probability of drawing a diamond after first drawing a queen

For P(A and B), two disjoint events:

1. The queen was a diamond
2. The queen was not a diamond

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Example 3: Drawing two cards from a deck of cards, the probability of drawing a diamond after first drawing a queen

For P(A and B), two disjoint events:

1. The queen was a diamond
2. The queen was not a diamond

$$\begin{aligned} P(A \text{ and } B) = {} & P(\text{second is diamond and first diamond queen}) \\ & + P(\text{second diamond and first queen of other suit}) \end{aligned}$$

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Example 3: Drawing two cards from a deck of cards, the probability of drawing a diamond after first drawing a queen

For P(A and B), two disjoint events:

1. The queen was a diamond
2. The queen was not a diamond

$$\begin{aligned} P(A \text{ and } B) &= P(\text{second is diamond and first diamond queen}) \\ &\quad + P(\text{second diamond and first queen of other suit}) \\ &= P(\text{second diamond} \mid \text{first diamond queen}) P(\text{first diamond queen}) \\ &\quad + P(\text{second diamond} \mid \text{first other queen}) P(\text{first other queen}) \end{aligned}$$

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Example 3: Drawing two cards from a deck of cards, the probability of drawing a diamond after first drawing a queen

For P(A and B), two disjoint events:

1. The queen was a diamond
2. The queen was not a diamond

$$\begin{aligned} P(A \text{ and } B) &= P(\text{second is diamond and first diamond queen}) \\ &+ P(\text{second diamond and first queen of other suit}) \\ &= P(\text{second diamond} \mid \text{first diamond queen}) P(\text{first diamond queen}) \\ &+ P(\text{second diamond} \mid \text{first other queen}) P(\text{first other queen}) \\ &= \frac{12}{51} \cdot \frac{1}{52} + \frac{13}{51} \cdot \frac{3}{52} \end{aligned}$$

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All together, $P(A|B) = \dfrac{\frac{12}{51} \cdot \frac{1}{52} + \frac{13}{51} \cdot \frac{3}{52}}{\frac{4}{52}} = \dfrac{1}{4}$

Question:

What is the probability of hitting a straight flush?